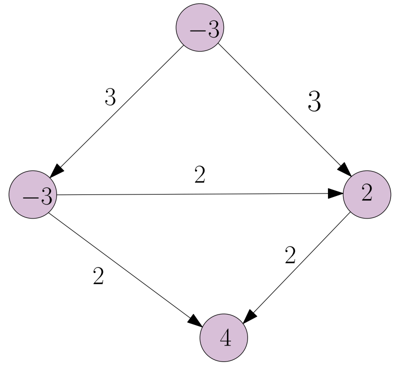
Part I

Circulations with demands

Circulations with demands



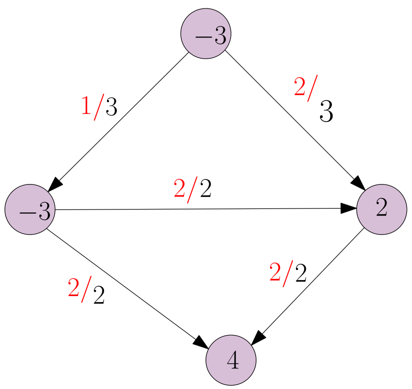
$\mathbf{G = (V, E)}$.
$\forall \boldsymbol{v} \in \mathbf{V}$ there is a ***demand*** $\boldsymbol{d_v}$:

- $\boldsymbol{d_v} > \mathbf{0}$: sink requiring $\boldsymbol{d_v}$ flow into this node.
- $\boldsymbol{d_v} < \mathbf{0}$: source with $-\boldsymbol{d_v}$ units of flow leaving it.
- $\boldsymbol{d_v} = \mathbf{0}$: regular node.

$\boldsymbol{S}$ set of source vertices
$\boldsymbol{T}$: set of sink vertices.

A circulation with demands: example

A valid circulation for the given instance



Definition: Circulation with demands

Definition

circulation with demands $\{\boldsymbol{d_v}\}$ is a function $\boldsymbol{f} : \mathbf{E(G)} \rightarrow \mathbb{R}^+$:

- Capacity condition: $\forall \boldsymbol{e} \in \boldsymbol{E}$ we have $\boldsymbol{f(e)} \leq \boldsymbol{c(e)}$.
- Conservation condition: $\forall \boldsymbol{v} \in \boldsymbol{V}$ we have $\boldsymbol{f^{in}(v)} - \boldsymbol{f^{out}(v)} = \boldsymbol{d_v}$.

Where:

1. $\boldsymbol{f^{in}(v)}$ flow into $\boldsymbol{v}$.
2. $\boldsymbol{f^{out}(v)}$: flow out of $\boldsymbol{v}$.

Problem

Is there a circulation that comply with the demand requirements?

Feasible circulation lemma

Lemma

If there is a feasible circulation with demands $\{d_v\}$, then $\sum_v d_v = 0$.

Proof.

Since it is a circulation, we have that $d_v = f^{in}(v) - f^{out}(v)$. Summing over all vertices: $\sum_v d_v = \sum_v f^{in}(v) - \sum_v f^{out}(v)$. The flow on every edge is summed twice, one with positive sign, one with negative sign. As such,

$$\sum_v d_v = \sum_v f^{in}(v) - \sum_v f^{out}(v) = 0,$$

which implies the claim. □

Computing circulations

∃ feasible circulation only if

$$D = \sum_{v, d_v > 0} d_v = \sum_{v, d_v < 0} -d_v.$$

Algorithm for computing circulation

(A) $\mathbf{G} = (\mathbf{V}, \mathbf{E})$: input flow network with demands on vertices.
(B) Check $D = \sum_{v, d_v > 0} d_v = \sum_{v, d_v < 0} -d_v$.
(C) Create super source s. Connect to all v with $d_v < 0$. Set capacity $(s \rightarrow v)$ to $-d_v$.
(D) Create super sink t. Connect to all vertices u with $d_u > 0$. Set capacity $(u \rightarrow t)$ to d_u.
(E) $\mathbf{H}$: new network flow. Compute max-flow f in $\mathbf{H}$ from s to t.
(F) If $|f| = D \implies$ ∃ valid circulation. Easy to recover.

Result: Circulations with demands

Theorem

∃ feasible circulation with demands $\{d_v\}$ in $\mathbf{G}$ $\iff$ max-flow in $\mathbf{H}$ has value D.
Integrality: *If all capacities and demands in $\mathbf{G}$ are integers, and there is a feasible circulation, then there is a feasible circulation that is integer valued.*

Part II

Circulations with demands and lower bounds

Circulations with demands and lower bounds

1. circulation and demands + for each edge a lower bound on flow.
2. $\forall e \in E(G)$: $\ell(e) \leq c(e)$.
3. Compute f such that $\forall e \quad \ell(e) \leq f(e) \leq c(e)$.
4. Be stupid! Consider flow: $\forall e \quad f_0(e) = \ell(e)$.
5. f_0 violates conservation of flow!
$$L_v = f_0^{in}(v) - f_0^{out}(v) = \sum_{e \text{ into } v} \ell(e) - \sum_{e \text{ out of } v} \ell(e).$$
6. If $L_v = d_v$, then no problem.
7. Fix-up demand: $\forall v \quad d'_v = d_v - L_v$.
Fix-up capacity: $c'(e) = c(e) - \ell(e)$.
8. G': new network w. new demands/capacities (no lower bounds!)
9. Compute circulation f' on G'.
$\implies$ The flow $f = f_0 + f'$, is a legal circulation,

Circulations with demands and lower bounds

Lemma

$\exists$ feasible circulation in G $\iff$ there is a feasible circulation in G'.
***Integrality**: If all numbers are integers $\implies$ $\exists$ integral feasible circulation.*

Proof.

Let f' be a circulation in G'. Let $f(e) = f_0(e) + f'(e)$.
Clearly, f satisfies the capacity condition in G, and the lower bounds.
$f^{in}(v) - f^{out}(v) = \sum_{e \text{ into } v}(\ell(e) + f'(e)) - \sum_{e \text{ out of } v}(\ell(e) + f'(e)) = L_v + (d_v - L_v) = d_v$.
f: valid circulation in G. Then $f'(e) = f(e) - \ell(e)$ is a valid circulation for G'. □

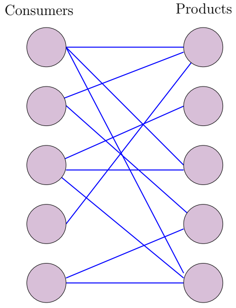
Part III

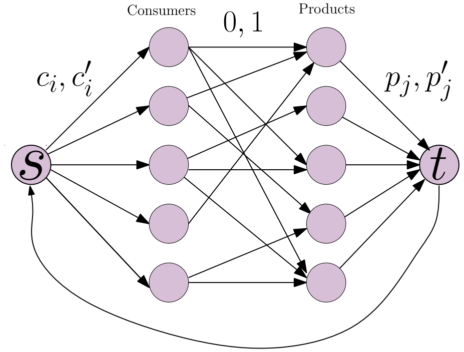
Applications

Survey design

1. Ask "Consumer i: what did you think of product j?"
2. ith consumer willing to answer between c_i to c'_i questions.
3. For each product j: at least p_j opinions, no more than p'_j opinions.
4. Full knowledge which consumers can be asked on which products.
5. Problem: How to assign questions to consumers?

Survey design...





Result...

Lemma

Given $\boldsymbol{n}$ *consumers and* $\boldsymbol{u}$ *products with their constraints* $\boldsymbol{c_1, c'_1, c_2, c'_2, \ldots, c_n, c'_n}$*,* $\boldsymbol{p_1, p'_1, \ldots, p_u, p'_u}$ *and a list of length* $\boldsymbol{m}$ *of which products where used by which consumers. An algorithm can compute a valid survey under these constraints, if such a survey exists, in time* $\boldsymbol{O((n+u)m^2)}$*.*