Approximating the Permanent

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KAIST (Spring '15)

Guest lecture for
KAIST CS 500
Graduate Algorithms
Wednesday, March 11, 2015

1. Permanent Definition
2. Random Matching
3. Random Perfect Matching

What is the Permanent?

3×3 *example:*

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

Determinant of A:

$$\det(A) = (aei + bfg + cdh) - (ceg + bdi + afh).$$

Permanent of A:

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In general, for a $n \times n$ matrix A, the determinant of A is

$$\det(A) = \sum_{\pi} \operatorname{sgn}(\pi) \prod_{i} A(i, \pi(i)),$$

where π ranges over all permutations of $\{1, \ldots, n\}$.

The permanent of A is

$$\operatorname{per}(A) = \sum_{\pi} \prod_{i} A(i, \pi(i))$$

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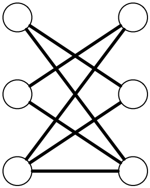

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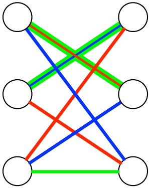

$$\begin{aligned} \mathrm{per}(A) &= aei + bfg + cdh + ceg + bdi + afh \\ &= bfg + cdh + bdi = 3 \end{aligned}$$

Applications

Some applications of the Permanent:

- *Statistical Physics:*
  - Dimer model of adsorption of diatomic molecules,
  - Ice-type models of crystal lattices with hydrogen bonds,
- *Computer Vision:* Tracking objects
- Number of graphs with specified degree sequence

What is the Permanent?

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$$\text{let } \mathcal{P} = \text{perfect matchings of } G.$$

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- Polynomial time algorithm for planar graphs [Kasteleyn '67]
- #P-complete for bipartite graphs [Valiant '79]
- FPRAS for counting *all* matchings [Jerrum-Sinclair '89]
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Agenda

Main tasks:

1. Count all matchings or generate a random matching.
2. Count perfect matchings or generate a random perfect matching.

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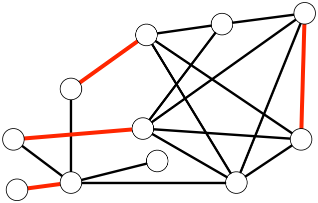
Random Matching

Undirected graph $G = (V, E)$:



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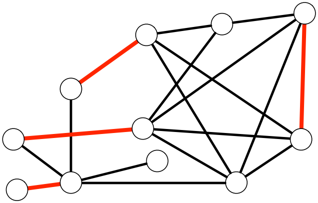

Matching = subset of vertex disjoint edges.

Let $\Omega =$ collection of all matchings of G (of all sizes).

Can we generate a matching uniformly at random from Ω?
in time polynomial in $n = |V|$?

Markov Chain for Matchings

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From a matching X_t the transition $X_t \to X_{t+1}$
is defined as follows:

1. Choose an edge $e = (v, w)$ uniformly at random from E.
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Symmetric and ergodic, hence:

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Mixing time $=$ How fast does it reach π?

Augmented Markov Chain for Matchings

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Example: $\Omega = \{1, 2, 3, 4\}$.
μ is uniform: $\mu(1) = \mu(2) = \mu(3) = \mu(4) = .25$.
And ν has: $\nu(1) = .5, \nu(2) = .1, \nu(3) = .15, \nu(4) = .25$.

$$d_{TV}(\mu, \nu) = \frac{1}{2}(.25 + .15 + .1 + 0) = .25$$

Mixing Time

Consider ergodic MC with states Ω, transition matrix P, and unique stationary distribution π.
For state $x \in \Omega$, time to mix from x:

$$T(x) = \min\{t : \mathrm{d}_{\mathrm{TV}}(\mathrm{P}^t(x, \cdot), \pi) \leq 1/4\}$$

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Rapidly mixing if $\mathrm{T}_{\mathrm{mix}} = \mathrm{poly}(n)$.

Relaxation time $\mathrm{T}_{\mathrm{rel}}$ = mixing time from a nice initial μ_0.

How to Bound Convergence Time

Underlying directed graph $H = (\Omega, E_{\mathrm{P}})$ of the Markov chain:

Vertices $=$ states Ω

Edges $= \{M \to M' : M, M' \in \Omega, \mathrm{P}(M, M') > 0\}$.

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For a set $S \subset \Omega$ where $\pi(S) \leq 1/2$ denote its conductance by:

$$\Phi(S) = \mathbf{Pr}\left(X_{t+1} \notin S \mid X_t \in S, X_t \sim \pi\right) = \sum_{M \in S, M' \in \overline{S}} \frac{\pi(M)\mathrm{P}(M, M')}{\pi(S)}$$

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$$\Omega(1/\Phi) = \mathrm{T}_{\mathrm{rel}} = O(1/\Phi^2).$$

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Hence, $\geq \frac{|S||\overline{S}|}{\rho\Omega} \geq \frac{|S|}{2\rho}$ transitions from S to $\overline{S}$. □

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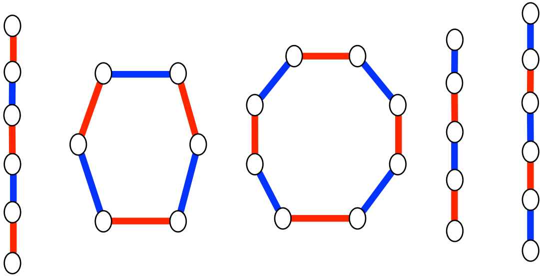
Look at their difference: $I \oplus F$.

Consists of alternating/augmenting paths and alternating cycles:



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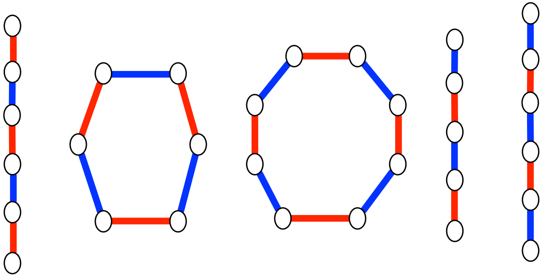

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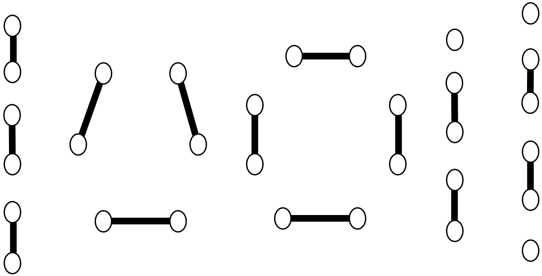

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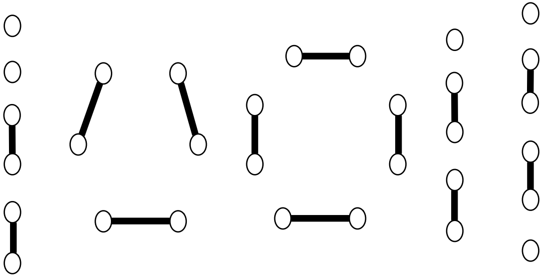

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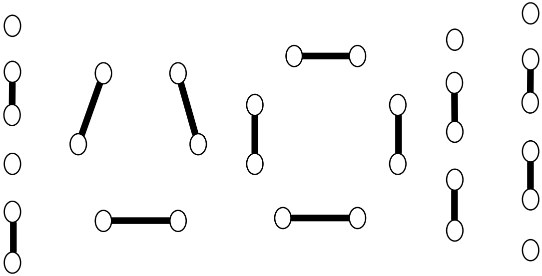

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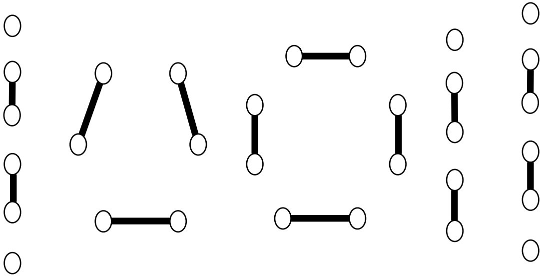

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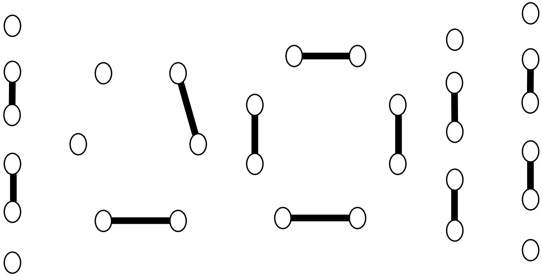

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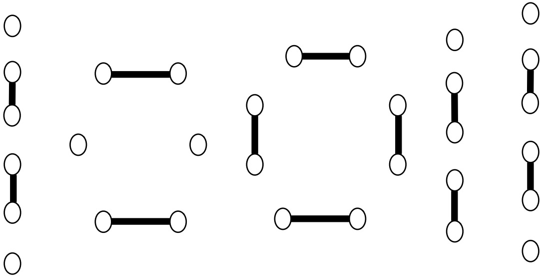

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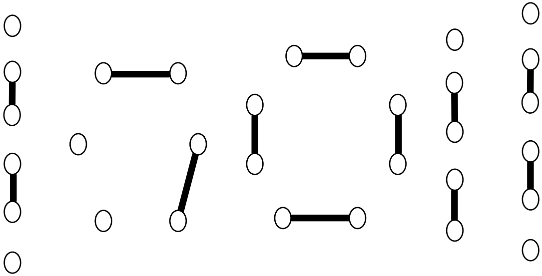

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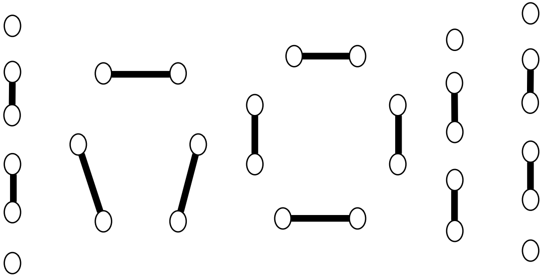

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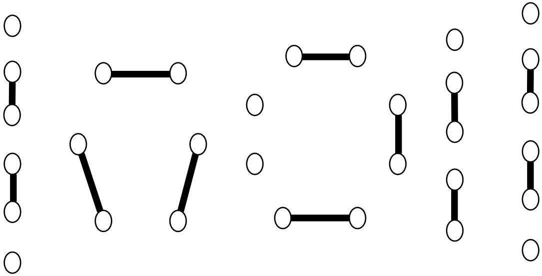

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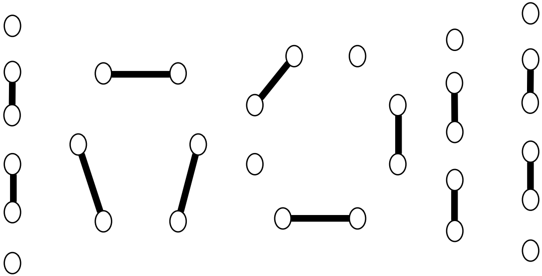

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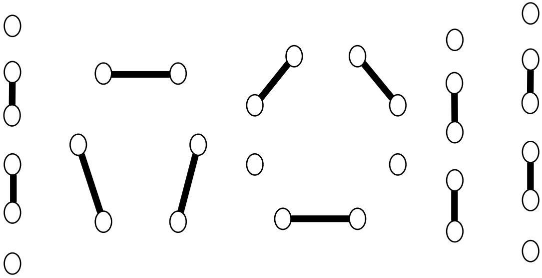

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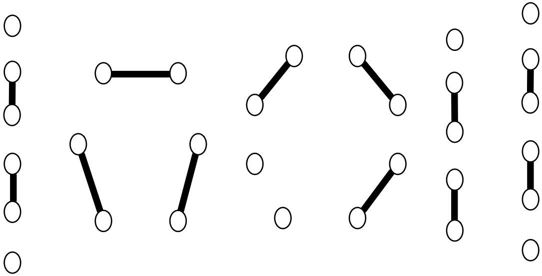

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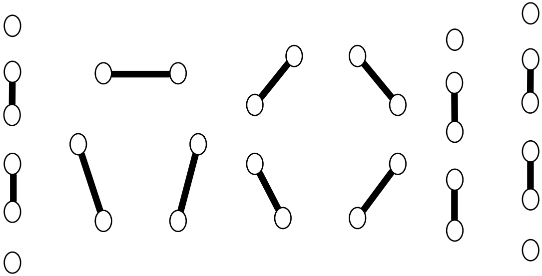

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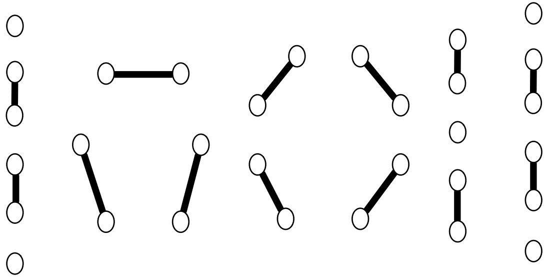

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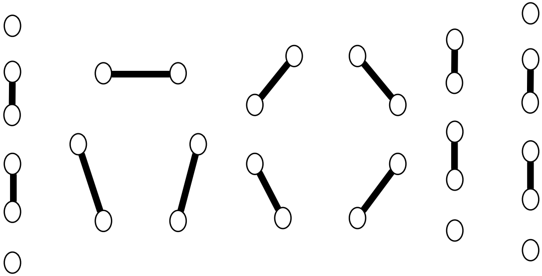

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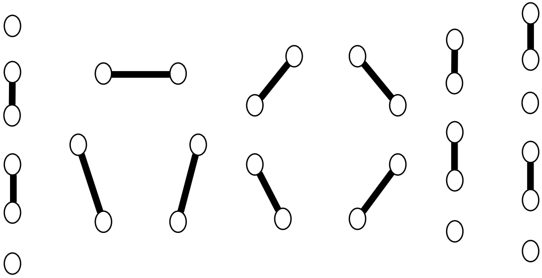

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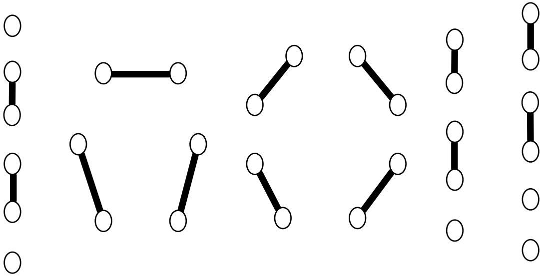

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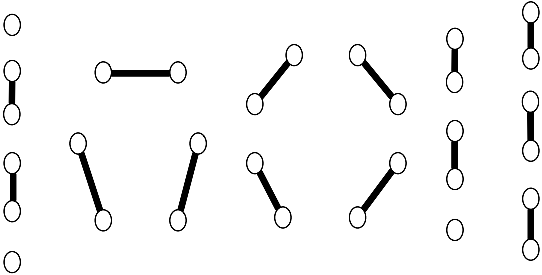

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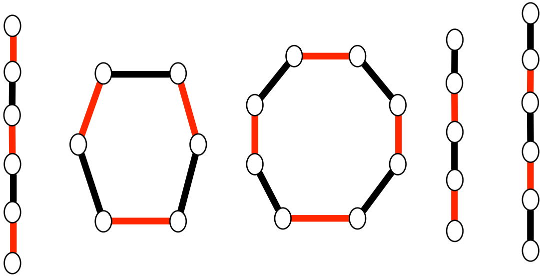

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Bounding Congestion

Consider a transition $T = M \to M'$.

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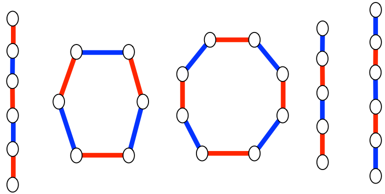
Easy to define η:

$$\eta_T(I, F) = (I \cap F) \bigcup (I \oplus F \setminus (M \cup M'))$$

Encoding

Example I and F:



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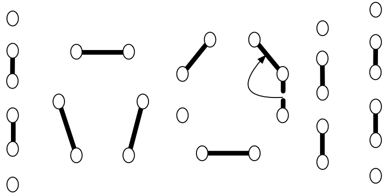

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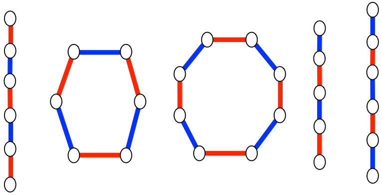

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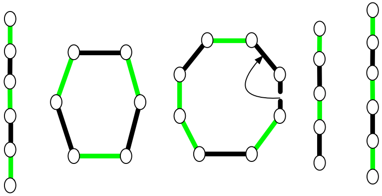

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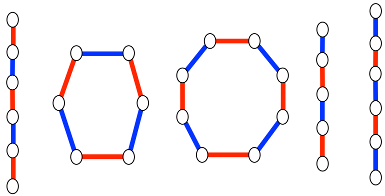

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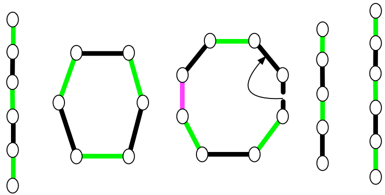

$$\eta_T(I,F) = (I \cap F) \bigcup (I \oplus F \setminus (M \cup M' \cup e_0))$$

where e_0 is the first edge of I in the current cycle.

1. Permanent Definition
2. Random Matching
3. Random Perfect Matching

First Idea for Markov Chain

For bipartite graph $G = (V, E)$ with $n + n$ vertices,

$$\text{let } \mathcal{P} = \text{ perfect matchings of } G.$$

Can we design a Markov chain only on $\mathcal{P}$?

What are the transitions?

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What are the transitions?

Enlarge the states: Near-perfect matchings:

$$\text{let } \mathcal{N} = \text{ matchings of } G \text{ with exactly 2 unmatched vertices.}$$

$$\text{Let } \Omega = \mathcal{P} \bigcup \mathcal{N}.$$

Run earlier Markov chain restricted to Ω.

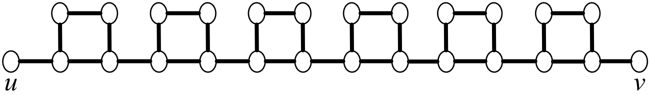
Markov Chain for Perfect Matchings

Consider an undirected bipartite graph $G = (V, E)$.
Let $\Omega = \mathcal{P} \bigcup \mathcal{N}$.

From a matching $X_t \in \Omega$ the transition $X_t \to X_{t+1}$ is defined by:

1. Choose an edge $e = (v, w)$ uniformly at random from E.
2. *Remove:* If $e \in X_t$ and $X_t \in \mathcal{P}$ then set $X_{t+1} = X_t \setminus \{e\}$.
3. *Add:* If v and w are unmatched in X_t then $X_{t+1} = X_t \bigcup \{e\}$.
4. *Slide:* If v is unmatched and w is matched (or vice-versa):
    1. Let (w, z) denote the matched edge.
    2. Set $X_{t+1} = X_t \bigcup (v, w) \setminus (w, z)$.
5. Otherwise, set $X_{t+1} = X_t$.

Bad Example



Key properties:

- $|\mathcal{P}| = 1$: Only 1 perfect matching
- $|\mathcal{N}| \geq 2^{n/4}$: if u and v unmatched then 2^s ways to complete where s is $\#$ of squares.

Conclusion:
Sampling from $\Omega = \mathcal{P} \bigcup \mathcal{N}$ may not help for sampling from $\mathcal{P}$.

Weights on Matchings

Assign matching $M \in \Omega$ a weight $w(M)$.

Add "Metropolis filter" to the Markov chain so that:

Stationary distribution $\pi(M) \propto w(M)$.

Choose weights so that:

1. $\pi(\mathcal{P}) = 1/\text{poly}(n)$ and every $P \in \mathcal{P}$ has the same weight.
2. Markov chain has mixing time poly(n).

Revised Markov Chain

Consider an undirected bipartite graph $G = (V, E)$.
Let $\Omega = \mathcal{P} \bigcup \mathcal{N}$.

From a matching $X_t \in \Omega$ the transition $X_t \to X_{t+1}$ is defined by:

1. Choose an edge $e = (v, y)$ uniformly at random from E.
2. *Remove:* If $e \in X_t$ and $X_t \in \mathcal{P}$ then set $X' = X_t \setminus \{e\}$.
3. *Add:* If v and y are unmatched in X_t then $X' = X_t \bigcup \{e\}$.
4. *Slide:* If v is unmatched and y is matched (or vice-versa):
    1. Let (y, z) denote the matched edge.
    2. Set $X' = X_t \bigcup (v, y) \setminus (y, z)$.
5. If X' is defined then:
   set $X_{t+1} = X'$ with probability $\min\{1, w(X')/w(X_t)\}$
6. Otherwise, set $X_{t+1} = X_t$.

Choice of Weights

Weight of matching $M \in \mathcal{P} \cup \mathcal{N}$ depends on unmatched vertices.

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$$\text{Note: } \sum_{P \in \mathcal{P}} w(P) = \sum_{N \in \mathcal{N}(u,v)} w(N) = |\mathcal{P}|$$

Hence: $\pi(\mathcal{P}) = \pi(\mathcal{N}(u, v)) = 1/(n^2 + 1)$.

Rapid Mixing

Key: for perfect matchings I, F, for $T = M \to M' \in \gamma_{I,F}$,

$$w(I)w(F) \geq w(M)w(\eta_T(I,F)).$$

Yields that Markov chain is rapidly mixing for these weights.

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Also, if use weights $w(u,v)$ where:

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then mixing time slows by a factor of 16.

Key: Can correct slightly wrong weights:

If $w(u,v) = \alpha\frac{|\mathcal{P}|}{|\mathcal{N}(u,v)|}$ then $\pi(\mathcal{N}(u,v)) = \alpha\pi(\mathcal{P})$ so:

- Generate many samples from π, and then
- Correct the weights $w(u,v)$.

Simulated Annealing Approach

Input bipartite graph $G = (L \cup R, E)$ captured by:
complete bipartite $K_{n,n}$ with edge activities for $y \in L, z \in R$:

$$\lambda(y,z) = \begin{cases} \lambda & \text{if } (y,z) \notin E \\ 1 & \text{if } (y,z) \in E \end{cases}$$

Slowly go from $\lambda = 1$ to $\lambda \approx 0$.

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Matching M of $K_{n,n}$ has activity: $\lambda(M) = \prod_{(y,z)\in M} \lambda(y,z)$.

$$\text{Redefine } w(u,v) = \frac{\lambda(\mathcal{P})}{\lambda(\mathcal{N}(u,v))}.$$

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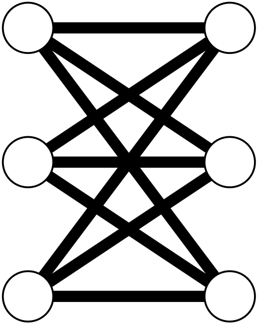
Algorithm:
Start with $\lambda = 1$ and $w(u,v) = n$ for all $u \in L, v \in R$.
Repeat until $\lambda < 1/n!$:

1. Set $\lambda = (1 - \frac{1}{2n})\lambda$.
2. Generate $O(n^2 \log n)$ samples from π.
3. Correct the weights $w(u,v)$ for all u, v.

Simulated Annealing Algorithm

Illustration of the algorithm:



$$weights = \begin{bmatrix} 3 & 3 & 3 \\ 3 & 3 & 3 \\ 3 & 3 & 3 \end{bmatrix}$$

1. Start at the complete bipartite graph
2. Slowly remove non-edges:
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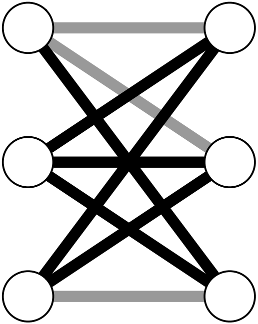

$$weights = \begin{bmatrix} 2.33 & 2.33 & 1.75 \\ 2.8 & 2.8 & 3.5 \\ 2.33 & 2.33 & 3.5 \end{bmatrix}$$

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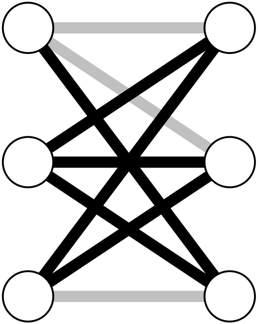

$$weights = \begin{bmatrix} 2.1 & 2.1 & 1.31 \\ 2.47 & 2.47 & 5.25 \\ 2.1 & 2.1 & 5.25 \end{bmatrix}$$

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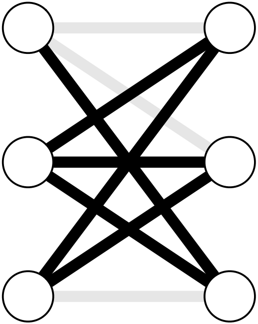

$$weights = \begin{bmatrix} 2.03 & 2.03 & 1.14 \\ 2.15 & 2.15 & 9.125 \\ 2.03 & 2.03 & 9.125 \end{bmatrix}$$

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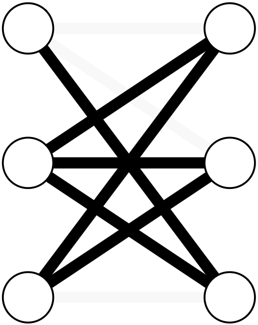

$$weights = \begin{bmatrix} 2.007 & 2.007 & 1.066 \\ 2.124 & 2.124 & 17.06 \\ 2.007 & 2.007 & 17.06 \end{bmatrix}$$

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The end

Thank you!