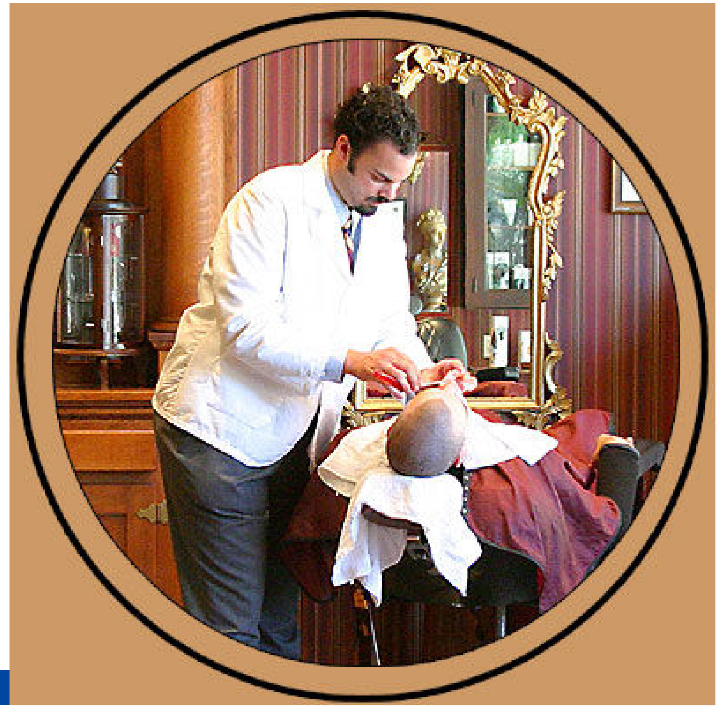
Barber‘s Theory

- Some men shave themselves.
- Others have the barber do it.
- **Hence the barber shaves all who don‘t shave themselves.**
- Really???
- So who is to shave the barber?
  - Case: He does shave himself.
  - Then he is a man not to be shaved by the barber! ↯
  - Case: He doesn‘t shave himself.
  - Then he is a man to be shaved by the barber! ↯



Diagonal Language

$D := \{ \langle P \rangle \mid$ program P does <u>not</u> terminate on input $\langle P \rangle \}$

Suppose D semi-decidable, by program P.

How does P behave on input $\langle P \rangle$?

Case: P terminates on $\langle P \rangle$.

Then $\langle P \rangle \notin D$ by definition of D.

But P semi-deciding D means,

P does <u>not</u> terminate on $\langle P \rangle \notin D$. ↯

Case: P does not terminate on $\langle P \rangle$.

Then $\langle P \rangle \in D$ by definition of D.

Similar contradiction. ↯

Rice's Theorem

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Examples: a) syntactical correctness ✓

b) $\{ \langle P \rangle : \langle P \rangle$ is ≤ 1000 characters long $\}$ ✓

c) $\{ \langle P \rangle : P$ makes ≤ 1000 steps on empty input $\}$ ✓

d) $\{ \langle P \rangle : P$ terminates on the empty input $\} = H$ ↯

e) $\{ \langle P \rangle : P$ semi-decides a non-empty set $\}$

Theorem (Rice-Myhill-Shapiro): Let $\mathbf{S}$ denote a <u>set</u> of subsets of $\{0,1\}^*$ and suppose some $L^+ \in \mathbf{S}$ and some $L^- \notin \mathbf{S}$ are semi-decidable.
Then, given P, it is undecidable whether the language semi-decided by P belongs to $\mathbf{S}$.

Rice‘s Theorem

d) $H = \{ \langle P \rangle : P$ terminates on the empty input $\}$

$\rightarrow \mathbf{S} := \{ L \subseteq \{0,1\}^* : L$ contains the empty string $\}$

e) $\{ \langle P \rangle : P$ semi-decides a non-empty set $\}$

$\rightarrow \mathbf{S} := \{ L \subseteq \{0,1\}^* : L$ is non-empty $\}$

„Any non-trivial property of the language semi-decided by a program is undecidable“

Theorem (Rice-Myhill-Shapiro): Let $\mathbf{S}$ denote a set of subsets of $\{0,1\}^*$ and suppose some $L^+ \in \mathbf{S}$ and some $L^- \notin \mathbf{S}$ are semi-decidable.

Then, given P, it is undecidable whether the language semi-decided by P belongs to $\mathbf{S}$.

Proof by Contradiction

Two cases: $L^0 := \emptyset \quad \in \mathbf{S}, \quad \notin \mathbf{S}$ (later)

Given P, decide "$\langle P \rangle \in H$" as follows:

- Construct from P a program Q which
  - first performs P (and doesn't terminate if P doesn't)
  - then invokes the program semi-deciding L^-.
- Q semi-decides $L^0 \in \mathbf{S}$ if $\langle P \rangle \notin H$ and $L^- \notin \mathbf{S}$ else.
- Case $L^0 \notin \mathbf{S}$: Let Q perform P then semi-decide L^+

Theorem (Rice-Myhill-Shapiro): Let $\mathbf{S}$ denote a <u>set</u> of subsets of $\{0,1\}^*$ and suppose some $L^+ \in \mathbf{S}$ and some $L^- \notin \mathbf{S}$ are semi-decidable.
Then, given P, it is undecidable whether the language semi-decided by P belongs to $\mathbf{S}$.

Oracle Computation

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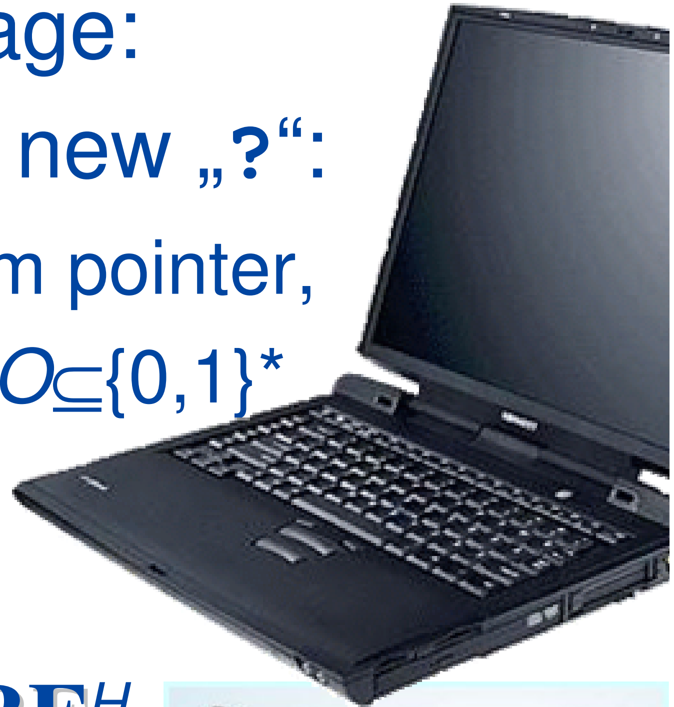
„*What if the Halting Problem were decidable?*“

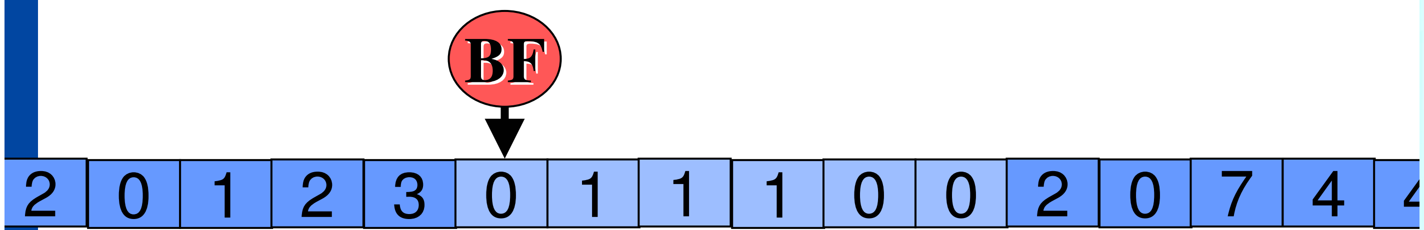
Extend **BF** programming language:

to commands `<>+-,.[]` add new „**?**“:

- considers binary sequence $\underline{\sigma}$ from pointer,
- sends $\underline{\sigma}$ to external, fixed ‘oracle‘ $O \subseteq \{0,1\}^*$
- and stores answer in current cell:
- 1 if $\underline{\sigma} \in O$, 0 if $\underline{\sigma} \notin O$.



Notation: $\mathbf{BF}^?$ and $\mathbf{BF}^O$; e.g. $\mathbf{BF}^H$



Power of Halting Oracle

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„What if the Halting Problem were decidable?“

BF^H can solve the Halting Problem: query the oracle!

Can also decide the Diagonal Language.

Can it decide <u>everything</u>?

- No: Still countably many **BF**^{H}-programs, uncountably many problems $L\subseteq\{0,1\}^*$.
- Specifically, cannot decide whether a given program terminates on <u>all</u> inputs!

$D = \{ \langle P\rangle \mid$ program P does <u>not</u> terminate on input $\langle P\rangle\}$

$H = \{ \langle P\rangle \mid$ program P terminates on the empty input$\}$

Totality:
even harder than the Halting Problem

$T := \{ \langle P \rangle \mid$ program P terminates on <u>all</u> inputs$\}$

Complement of T is semi-decidable relative to H:

Given P,

- for each input $z \in \{0,1\}^*$:
  - construct program $Q = Q(P,z)$ which
    - ignores its input and behaves like P on z.
  - Query „$\langle Q \rangle \in H$?“ and terminate if answer „`no`“.

Theorem: T is <u>not</u> semi-decidable relative to H !

$H = \{ \langle P \rangle \mid$ program P terminates on the empty input$\}$

Relativized Undecidability

Fix arbitrary oracle $O \subseteq \{0,1\}^*$
and consider the relativized diagonal language
$D^O := \{\langle P^? \rangle \mid$ program P^O doesn't terminate on input $\langle P^? \rangle\}$
is not semi-decidable by a **BF**^{O} program:

- Suppose P^O semi-decides D^O.
- Case $\langle P^? \rangle \in D^O$: … contradiction.
- Case $\langle P^? \rangle \notin D^O$: … contradiction.

Therefore the relativized Halting problem
$H^O := \{\langle P^? \rangle \mid$ program P^O terminates on the empty input$\}$
is not decidable by a **BF**^{O} program.

$D = \{ \langle P \rangle \mid$ program P does <u>not</u> terminate on input $\langle P \rangle\}$

$H = \{ \langle P \rangle \mid$ program P terminates on the empty input$\}$

Arithmetical Hierarchy

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$H^O := \{\langle P^?\rangle \mid \text{program } P^O \text{ terminates on the empty input}\}$
is not decidable by a $\mathbf{BF}^O$ program

- $\emptyset$ decidable by $\mathbf{BF}$. ✓
- $\emptyset' := H$ semi-decidable
  - but not decidable by $\mathbf{BF}$
  - yet decidable by $\mathbf{BF}^H$
- $\emptyset'' := H^H = H^{\emptyset'}$ semi-decidable by $\mathbf{BF}^H$
  - but not decidable by $\mathbf{BF}^H$
  - yet decidable by $\mathbf{BF}^{H^H} = \mathbf{BF}^{\emptyset''}$
- $\emptyset''' := H^{\emptyset''}$ semi-decidable by $\mathbf{BF}^{\emptyset''}$
  - but not decidable by $\mathbf{BF}^{\emptyset''}$
  - yet decidable by $\mathbf{BF}^{\emptyset'''}$
- $\emptyset'''' =: \emptyset^{(4)}$, $\emptyset^{(5)}$, ……