Satisfiability

Consider a set of boolean variables $x_1, x_2, \dots x_n$.

A literal is either a boolean variable x_i or its negation $\neg x_i$.

A clause is a disjunction of literals.
For example, $x_1 \vee x_2 \vee \neg x_4$ is a clause.

A formula in conjunctive normal form (CNF) is propositional formula which is a conjunction of clauses:

$$(x_1 \vee x_2 \vee \neg x_4) \wedge (x_2 \vee \neg x_3) \wedge x_5$$

A CNF formula such that every clause has exactly 3 literals is a 3CNF formula.

$$(x_1 \vee x_2 \vee \neg x_4) \wedge (x_2 \vee \neg x_3 \vee x_1)$$

SAT and 3SAT

SAT:
Instance: A CNF formula ϕ.
Question: Is there a truth assignment to the variables of ϕ such that ϕ evaluates to true?

3SAT:
Instance: A 3CNF formula ϕ.
Question: Is there a truth assignment to the variables of ϕ such that ϕ evaluates to true?

$(x_1 \vee x_2 \vee \neg x_4) \wedge (x_2 \vee \neg x_3) \wedge x_5$ is satisfiable; take $x_1, x_2, \dots x_5$ to be all true

$(x_1 \vee \neg x_2) \wedge (\neg x_1 \vee x_2) \wedge (\neg x_1 \vee \neg x_2) \wedge (x_1 \vee x_2)$ is not satisfiable.

Importance of Satisfiability

- SAT and 3SAT are basic constraint satisfaction problems.
- Many different problems can be reduced to SAT.
- Arise naturally in many applications involving hardware and software verification and correctness.
- As we will see, it is a fundamental problem in the theory of NP-completeness.

SAT and 3SAT

Clearly we have 3SAT $\leq$ SAT

But we also have SAT $\leq$ 3SAT

Given φ a CNF formula we create a 3CNF formula φ' such that
- φ is satisfiable iff φ' is satisfiable.
- φ' can be constructed from φ in time polynomial in $|\varphi|$.

Idea: if a clause of φ is not of length 3, replace it with several clauses of length exactly 3.

One and two literals

Clause with 2 literals: Let $c = \ell_1 \vee \ell_2$.
Let u be a new variable and let

$$c' = (\ell_1 \vee \ell_2 \vee u) \wedge (\ell_1 \vee \ell_2 \vee \neg u).$$

Then c is satisfiable iff c' is satisfiable

Clause with 1 literal: Let $c = \ell_1$.
Let u and v be new variables and let

$$c' = (\ell_1 \vee u \vee v) \wedge (\ell_1 \vee u \vee \neg v) \wedge (\ell_1 \vee \neg u \vee v) \wedge (\ell_1 \vee \neg u \vee \neg v).$$

Then c is satisfiable iff c' is satisfiable

More than 3 literals

Let $c = \ell_1 \vee \cdots \vee \ell_k$.
Let $u_1, \ldots u_{k-3}$ be new variables. Consider

$$\begin{aligned} c' = &(\ell_1 \vee \ell_2 \vee u_1) \wedge \\ &(\ell_3 \vee \neg u_1 \vee u_2) \wedge \\ &(\ell_4 \vee \neg u_2 \vee u_3) \wedge \\ &\cdots \wedge \\ &(\ell_{k-2} \vee \neg u_{k-4} \vee u_{k-3}) \wedge \\ &(\ell_{k-1} \vee \ell_k \vee \neg u_{k-3}). \end{aligned}$$

c is satisfiable if and only if c' is satisfiable.

2SAT

What about 2SAT?

It can be solved in polynomial time (even linear time).

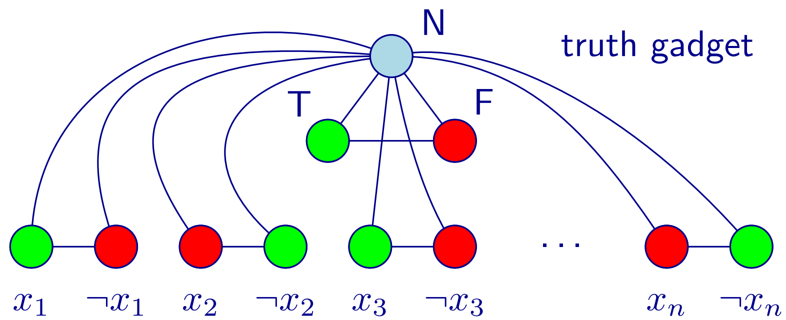
Compare:

2COLORING	3COLORING
2SAT	3SAT
Easy	Hard

3SAT $\leq$ 3COLORING

Reduction converts a formula ϕ in 3CNF into a graph G.

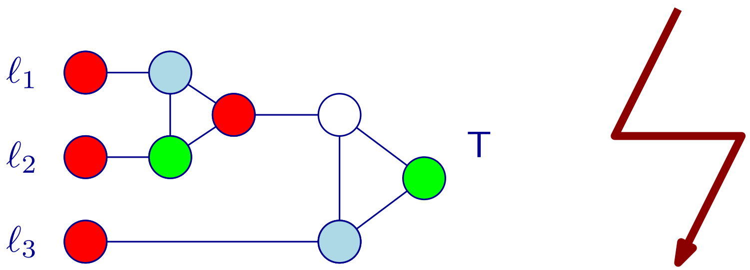
ϕ is satisfiable iff G can be colored with three colors.



We have one variable gadget for each variable. x_i and $\neg x_i$ must have different colors.
All x_i and $\neg x_i$ vertices are connected to N, so they cannot be blue. So either x_i is green (and $\neg x_i$ red), or x_i is red (and $\neg x_i$ green).

Clause gadgets

For each clause $c = \ell_1 \vee \ell_2 \vee \ell_3$ we build a gadget that can be colored iff at least one of the ℓ_i nodes is green.



Clause gadgets

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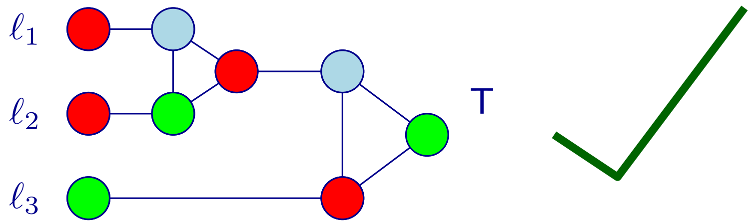
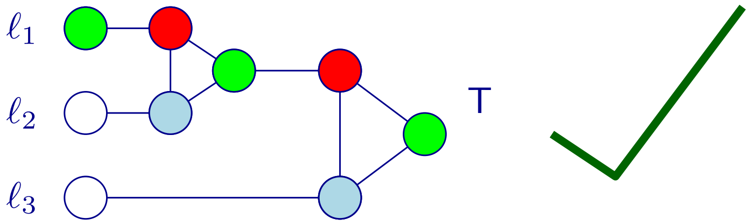


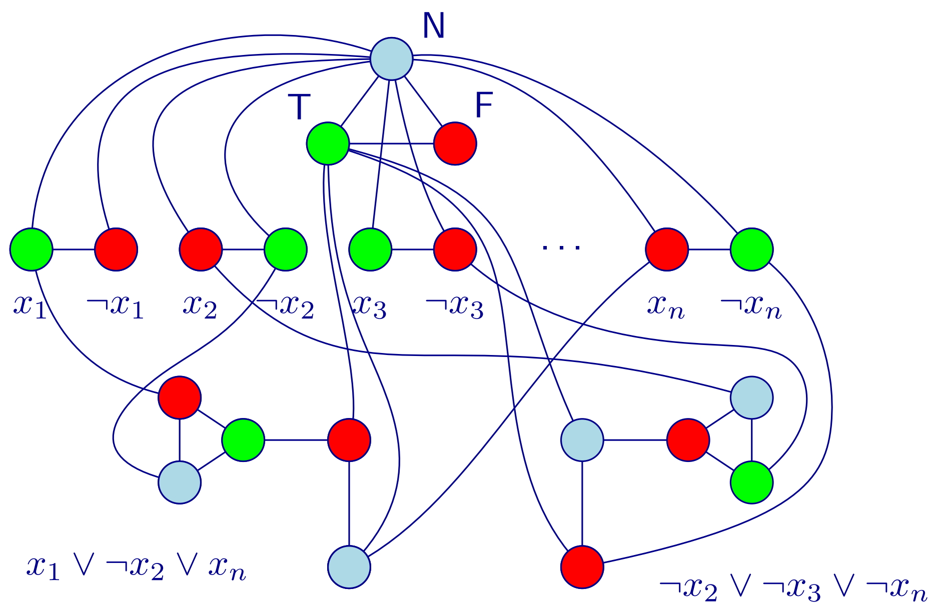
Clause gadgets

For each clause $c = \ell_1 \vee \ell_2 \vee \ell_3$ we build a gadget that can be colored iff at least one of the ℓ_i nodes is green.

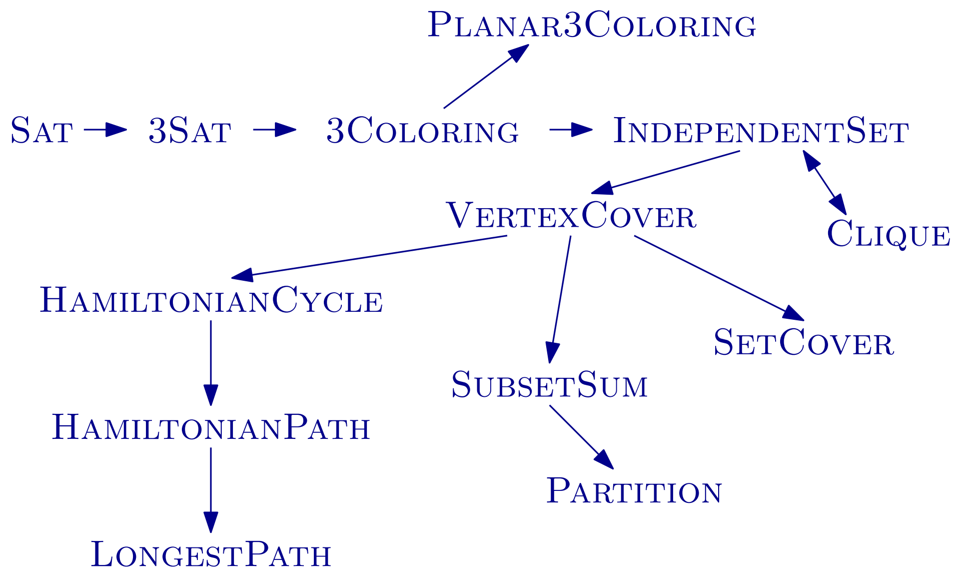


Complete construction

We create a clause gadget for each clause.



Reduction Summary



So SAT $\leq$ X, for all these problems X.