Graph Coloring

k-COLORING:

Instance: A graph G.

Question: Can the vertices of G be colored with k colors such that no two vertices of the same color are adjacent?

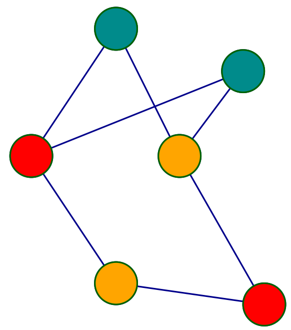
(Note that k is part of the problem, not an input.)

$$k\text{-COLORING} \leq (k+1)\text{-COLORING}$$

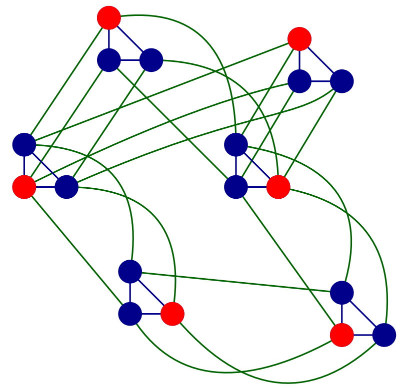
Lemma: 2-COLORING can be solved in linear time.

Independent Set and Coloring

3-COLORING $\leq$ INDEPENDENTSET



n vertices



$3n$ vertices, $k = n$

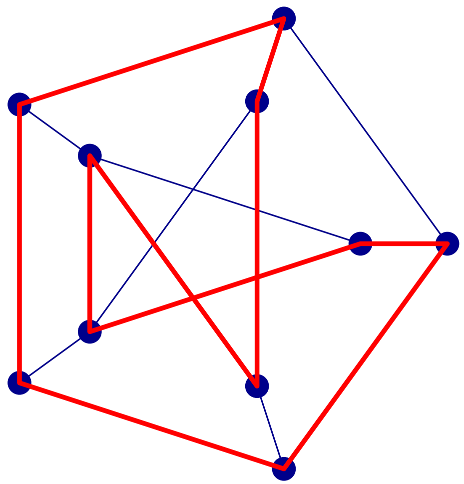
3-Coloring $\Leftrightarrow$ independent set of size n

Hamiltonian Cycle

HAMILTONIANCYCLE:

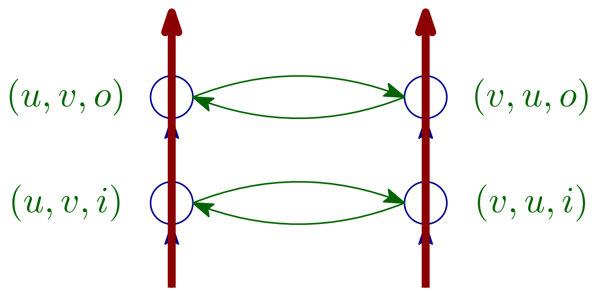
Instance: A directed graph G.

Question: Does G contain a Hamiltonian cycle (a cycle that visits every vertex exactly once)?



Vertex Cover and Hamiltonian Cycle

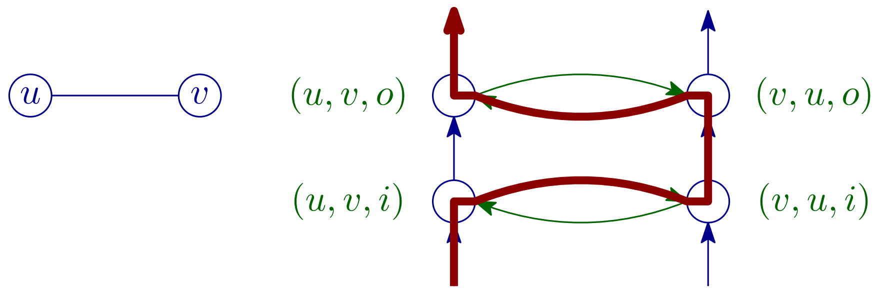
VERTEXCOVER $\leq$ HAMILTONIANCYCLE



There are three possible ways for a Hamiltonian cycle to visit these four vertices.

Vertex Cover and Hamiltonian Cycle

VertexCover $\leq$ HamiltonianCycle



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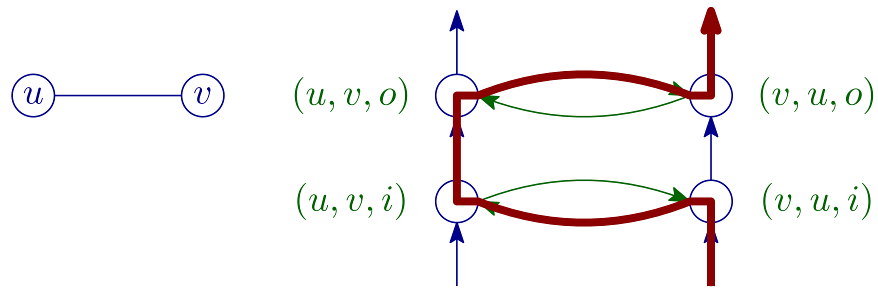
Vertex Cover and Hamiltonian Cycle

VertexCover $\leq$ HamiltonianCycle



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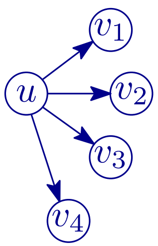
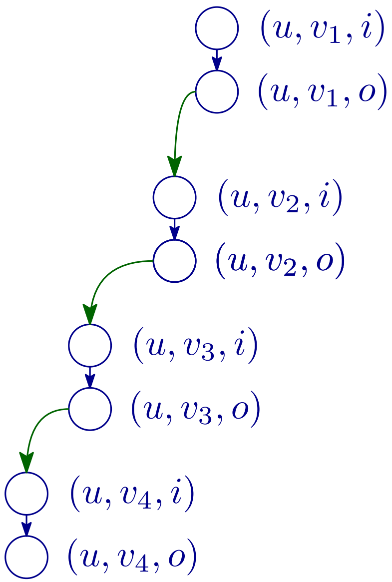
Vertices of G



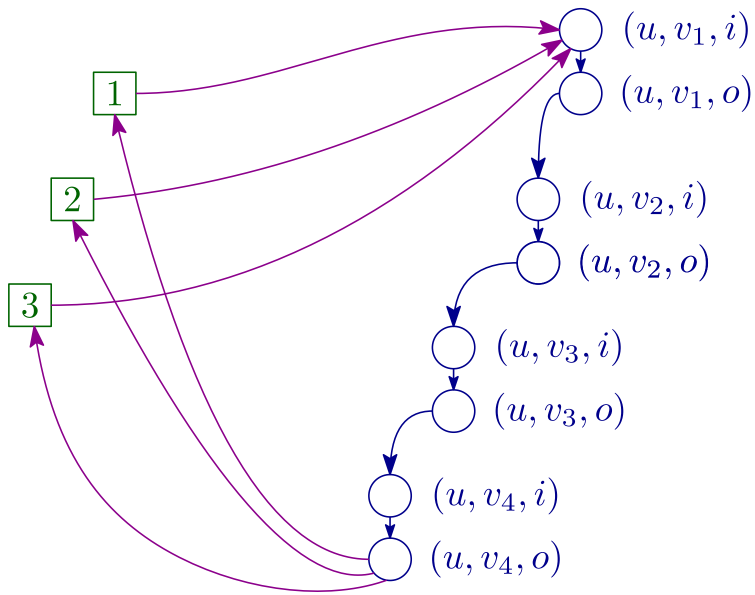
Vertex chain of u



Cover vertices

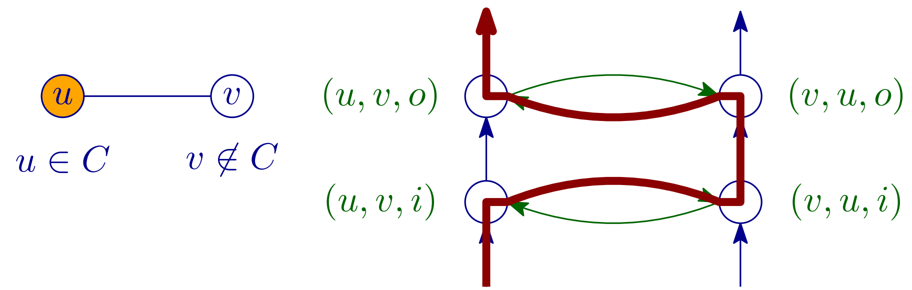
k cover vertices connected to the ends of all vertex chains.

$k = 3$



G has a vertex cover of size k if and only if G' has a Hamiltonian path.

Let $u_1, u_2, \ldots, u_k$ be the vertices of the vertex cover C. Then the Hamiltonian path starts in cover vertex 1, visits the vertex chain of u_1, goes to cover vertex 2, visits the vertex chain of u_2, and so on, until returning to cover vertex 1.



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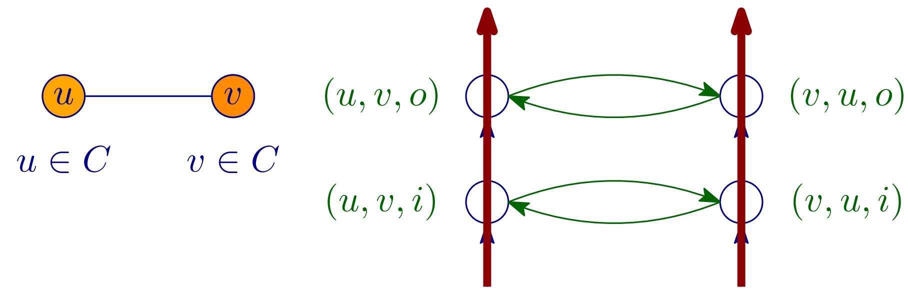
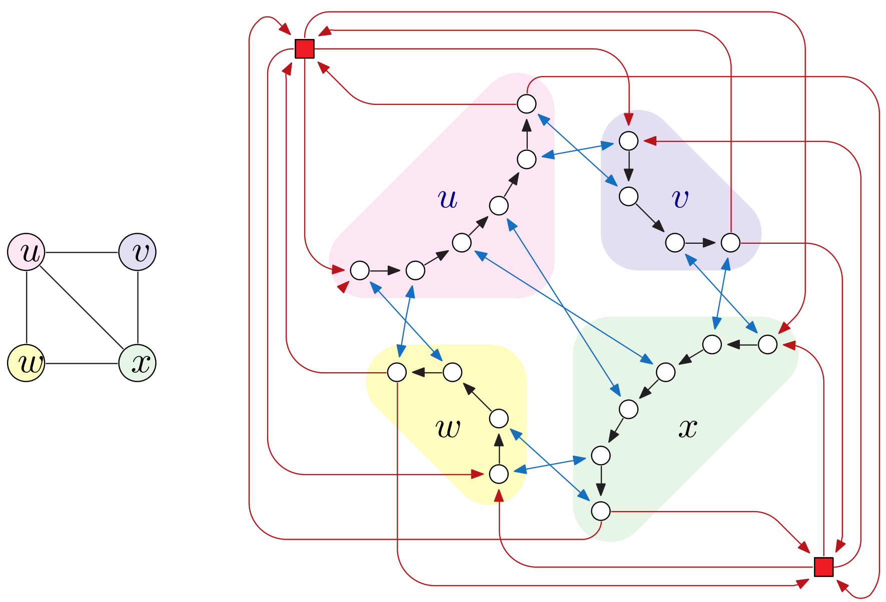
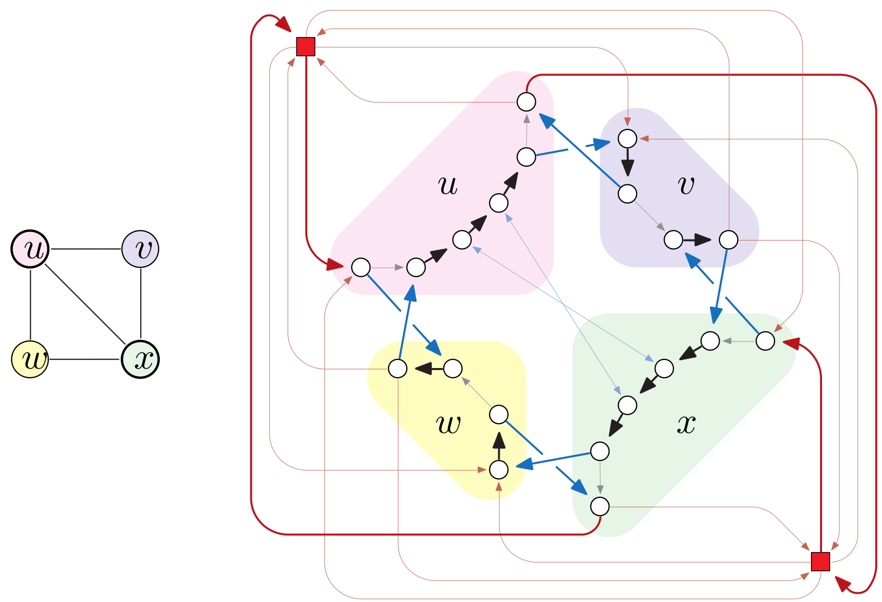


Example



Example Cover



SUBSETSUM:
Instance: A set X of positive integers and an integer t.
Question: Does X have a subset whose elements sum to t?

$$\text{VERTEXCOVER} \leq \text{SUBSETSUM}$$

Number edges from 0 to $m-1$.

Our set X contains $b_i = 4^i$ for each edge i, and a_v for each vertex v:

$$a_v = 4^m + \sum_{i \in \Delta(v)} 4^i.$$

The target sum t is

$$t = k \cdot 4^m + \sum_{i=0}^{m-1} 2 \cdot 4^i.$$

3-COLORING ≤ INDEPENDENTSET

3-COLORING ≤ PLANAR3COLORING

INDEPENDENTSET ≤ VERTEXCOVER

INDEPENDENTSET ≤ CLIQUE

VERTEXCOVER ≤ SETCOVER

VERTEXCOVER ≤ HAMILTONIANCYCLE

VERTEXCOVER ≤ SUBSETSUM

SUBSETSUM ≤ PARTITION

HAMILTONIANCYCLE ≤ HAMILTONIANPATH

HAMILTONIANPATH ≤ LONGESTPATH