Part I

Weighted vertex cover

Weighted vertex cover

Weighted Vertex Cover problem

$\mathbf{G = (V, E)}$.

Each vertex $\mathbf{v} \in \mathbf{V}$: cost $\mathbf{c_v}$.

Compute a vertex cover of minimum cost.

1. vertex cover: subset of vertices $\mathbf{V}$ so each edge is covered.
2. **NP-Hard**
3. ...unweighted **Vertex Cover** problem.
4. ... write as an integer program (IP):
5. $\forall \mathbf{v} \in \mathbf{V}$: $x_v = 1 \iff \mathbf{v}$ in the vertex cover.
6. $\forall \mathbf{vu} \in \mathbf{E}$: covered. $\implies x_v \vee x_u$ **true**. $\implies$ $x_v + x_u \geq 1$.
7. minimize total cost: $\min \sum_{v \in V} x_v c_v$.

Weighted vertex cover

$$
\begin{aligned}
\min \quad & \sum_{v \in V} c_v x_v, & & \\
\text{such that} \quad & x_v \in \{0,1\} & \forall v \in V & \qquad (1) \\
& x_v + x_u \geq 1 & \forall vu \in E. &
\end{aligned}
$$

1. ... **NP-Hard**.
2. relax the integer program.
3. allow x_v get values $\in [0,1]$.
4. $x_v \in \{0,1\}$ replaced by $0 \leq x_v \leq 1$. The resulting LP is

$$
\begin{aligned}
\min \quad & \sum_{v \in V} c_v x_v, & \\
\text{s.t.} \quad & 0 \leq x_v & \forall v \in V, \\
& x_v \leq 1 & \forall v \in V, \\
& x_v + x_u \geq 1 & \forall vu \in E.
\end{aligned}
$$

Weighted vertex cover – rounding the LP

1. Optimal solution to this LP: $\widehat{x_v}$ value of var X_v, $\forall v \in V$.
2. optimal value of LP solution is $\widehat{\alpha} = \sum_{v \in V} c_v \widehat{x_v}$.
3. optimal integer solution: x_v^I, $\forall v \in V$ and α^I.
4. **Any valid solution to IP is valid solution for LP!**
5. $\widehat{\alpha} \leq \alpha^I$.
   Integral solution not better than LP.
6. Got fractional solution (i.e., values of $\widehat{x_v}$).
7. Fractional solution is better than the optimal cost.
8. Q: How to turn fractional solution into a (valid!) integer solution?
9. Called ***rounding***.

How to round?

1. consider vertex $\mathbf{v}$ and fractional value $\widehat{x_v}$.
2. If $\widehat{x_v} = 1$ then include in solution!
3. If $\widehat{x_v} = 0$ then do not include in solution.
4. if $\widehat{x_v} = 0.9 \implies$ LP considers $\mathbf{v}$ as being $\mathbf{0.9}$ useful.
5. The LP puts its money where its belief is...
6. ...$\widehat{\alpha}$ value is a function of this "belief" generated by the LP.
7. Big idea: Trust LP values as guidance to usefulness of vertices.
8. Pick all vertices $\geq$ threshold of usefulness according to LP.
9. $S = \left\{ v \;\middle|\; \widehat{x_v} \geq 1/2 \right\}$.
10. **Claim**: S a valid vertex cover, and cost is low.
11. Indeed, edge cover as: $\forall vu \in E$ have $\widehat{x_v} + \widehat{x_u} \geq 1.$
12. $\widehat{x_v}, \widehat{x_u} \in (0, 1)$
    $\implies \widehat{x_v} \geq 1/2$ or $\widehat{x_u} \geq 1/2$.
    $\implies v \in S$ or $u \in S$ (or both).
    $\implies S$ covers all the edges of G.

Cost of solution

Cost of S:

$$c_S = \sum_{v \in S} c_v = \sum_{v \in S} 1 \cdot c_v \leq \sum_{v \in S} 2\widehat{x_v} \cdot c_v \leq 2 \sum_{v \in V} \widehat{x_v} c_v = 2\widehat{\alpha} \leq 2\alpha',$$

since $\widehat{x_v} \geq 1/2$ as $v \in S$.
α' is cost of the optimal solution $\implies$

Theorem
*The **Weighted Vertex Cover** problem can be 2-approximated by solving a single LP. Assuming computing the LP takes polynomial time, the resulting approximation algorithm takes polynomial time.*

The lessons we can take away

Or not - boring, boring, boring.

1. Weighted vertex cover is simple, but resulting approximation algorithm is non-trivial.
2. Not aware of any other **2**-approximation algorithm does not use LP. (For the weighted case!)
3. Solving a ***relaxation*** of an optimization problem into a LP provides us with insight.
4. But... have to be creative in the rounding.