

Part I

Weighted vertex cover

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Weighted Vertex Cover problem

$G = (V, E)$.

Each vertex $v \in V$: cost c_v .

Compute a vertex cover of minimum cost.

1. vertex cover: subset of vertices V so each edge is covered.
2. **NP-Hard**
3. ...unweighted **Vertex Cover** problem.
4. ... write as an integer program (IP):
5. $\forall v \in V: x_v = 1 \iff v$ in the vertex cover.
6. $\forall vu \in E$: covered. $\implies x_v \vee x_u$ true. $\implies x_v + x_u \geq 1$.
7. minimize total cost: $\min \sum_{v \in V} x_v c_v$.

Weighted vertex cover

$$\begin{aligned} \min \quad & \sum_{v \in V} c_v x_v, \\ \text{such that} \quad & x_v \in \{0, 1\} \quad \forall v \in V \quad (1) \\ & x_v + x_u \geq 1 \quad \forall vu \in E. \end{aligned}$$

1. ... **NP-Hard**.
2. relax the integer program.
3. allow x_v get values $\in [0, 1]$.
4. $x_v \in \{0, 1\}$ replaced by $0 \leq x_v \leq 1$. The resulting **LP** is

$$\begin{aligned} \min \quad & \sum_{v \in V} c_v x_v, \\ \text{s.t.} \quad & 0 \leq x_v \quad \forall v \in V, \\ & x_v \leq 1 \quad \forall v \in V, \\ & x_v + x_u \geq 1 \quad \forall vu \in E. \end{aligned}$$

Weighted vertex cover – rounding the LP

1. Optimal solution to this **LP**: \hat{x}_v value of var $X_v, \forall v \in V$.
2. optimal value of **LP** solution is $\hat{\alpha} = \sum_{v \in V} c_v \hat{x}_v$.
3. optimal integer solution: $x_v^I, \forall v \in V$ and α^I .
4. **Any valid solution to IP is valid solution for LP!**
5. $\hat{\alpha} \leq \alpha^I$.
Integral solution not better than **LP**.
6. Got fractional solution (i.e., values of \hat{x}_v).
7. Fractional solution is better than the optimal cost.
8. Q: How to turn fractional solution into a (valid!) integer solution?
9. Called **rounding**.

How to round?

1. consider vertex \mathbf{v} and fractional value $\widehat{x}_{\mathbf{v}}$.
2. If $\widehat{x}_{\mathbf{v}} = 1$ then include in solution!
3. If $\widehat{x}_{\mathbf{v}} = 0$ then do **not** include in solution.
4. if $\widehat{x}_{\mathbf{v}} = 0.9 \implies$ LP considers \mathbf{v} as being **0.9** useful.
5. The LP puts its money where its belief is...
6. ... $\widehat{\alpha}$ value is a function of this "belief" generated by the LP.
7. **Big idea:** Trust LP values as guidance to usefulness of vertices.
8. Pick all vertices \geq threshold of usefulness according to LP.
9. $\mathbf{S} = \{\mathbf{v} \mid \widehat{x}_{\mathbf{v}} \geq 1/2\}$.
10. **Claim:** \mathbf{S} a valid vertex cover, and cost is low.
11. Indeed, edge cover as: $\forall \mathbf{vu} \in \mathbf{E}$ have $\widehat{x}_{\mathbf{v}} + \widehat{x}_{\mathbf{u}} \geq 1$.
12. $\widehat{x}_{\mathbf{v}}, \widehat{x}_{\mathbf{u}} \in (0, 1)$
 - $\implies \widehat{x}_{\mathbf{v}} \geq 1/2$ or $\widehat{x}_{\mathbf{u}} \geq 1/2$.
 - $\implies \mathbf{v} \in \mathbf{S}$ or $\mathbf{u} \in \mathbf{S}$ (or both).
 - $\implies \mathbf{S}$ covers all the edges of \mathbf{G} .

The lessons we can take away

Or not - boring, boring, boring.

1. Weighted vertex cover is simple, but resulting approximation algorithm is non-trivial.
2. Not aware of any other 2-approximation algorithm does not use LP. (For the weighted case!)
3. Solving a **relaxation** of an optimization problem into a LP provides us with insight.
4. But... have to be creative in the rounding.

Cost of solution

Cost of \mathbf{S} :

$$c_{\mathbf{S}} = \sum_{\mathbf{v} \in \mathbf{S}} c_{\mathbf{v}} = \sum_{\mathbf{v} \in \mathbf{S}} 1 \cdot c_{\mathbf{v}} \leq \sum_{\mathbf{v} \in \mathbf{S}} 2\widehat{x}_{\mathbf{v}} \cdot c_{\mathbf{v}} \leq 2 \sum_{\mathbf{v} \in \mathbf{V}} \widehat{x}_{\mathbf{v}} c_{\mathbf{v}} = 2\widehat{\alpha} \leq 2\alpha'$$

since $\widehat{x}_{\mathbf{v}} \geq 1/2$ as $\mathbf{v} \in \mathbf{S}$.

α' is cost of the optimal solution \implies

Theorem

The **Weighted Vertex Cover** problem can be 2-approximated by solving a single LP. Assuming computing the LP takes polynomial time, the resulting approximation algorithm takes polynomial time.