Duality by Example

$$\begin{array}{rl} \max & z = 4x_1 + x_2 + 3x_3 \\ \text{s.t.} & x_1 + 4x_2 \quad \leq 1 \\ & 3x_1 - x_2 + x_3 \leq 3 \\ & x_1, x_2, x_3 \geq 0 \end{array}$$

1. η: maximal possible value of target function.
2. Any feasible solution $\Rightarrow$ a lower bound on η.
3. In above: $x_1 = 1, x_2 = x_3 = 0$ is feasible, and implies $z = 4$ and thus $\eta \geq 4$.
4. $x_1 = x_2 = 0$, $x_3 = 3$ is feasible $\implies \eta \geq z = 9$.
5. How close this solution is to opt? (i.e., η)
6. If very close to optimal – might be good enough. Maybe stop?

Duality by Example: II

$$\begin{array}{rl} \max & z = 4x_1 + x_2 + 3x_3 \\ \text{s.t.} & x_1 + 4x_2 \quad \leq 1 \\ & 3x_1 - x_2 + x_3 \leq 3 \\ & x_1, x_2, x_3 \geq 0 \end{array}$$

1. Add the first inequality (multiplied by 2) to the second inequality (multiplied by 3):

$$\begin{array}{r} 2(\ x_1 + 4x_2 \qquad) \leq 2(1) \\ +3(3x_1 - x_2 + x_3) \leq 3(3). \end{array}$$

2. The resulting inequality is

$$11x_1 + 5x_2 + 3x_3 \leq 11. \qquad (1)$$

Duality by Example: II

$$\begin{array}{rl} \max & z = 4x_1 + x_2 + 3x_3 \\ \text{s.t.} & x_1 + 4x_2 \quad \leq 1 \\ & 3x_1 - x_2 + x_3 \leq 3 \\ & x_1, x_2, x_3 \geq 0 \end{array}$$

1. got $11x_1 + 5x_2 + 3x_3 \leq 11$.
2. inequality must hold for any feasible solution of L.
3. Objective: $z = 4x_1 + x_2 + 3x_3$ and x_1,x_2 and x_3 are all non-negative.
4. Inequality above has larger coefficients than objective (for corresponding variables)
5. For any feasible solution: $z = 4x_1 + x_2 + 3x_3 \leq 11x_1 + 5x_2 + 3x_3 \leq 11$,

Duality by Example: III

$$\begin{array}{rl} \max & z = 4x_1 + x_2 + 3x_3 \\ \text{s.t.} & x_1 + 4x_2 \quad \leq 1 \\ & 3x_1 - x_2 + x_3 \leq 3 \\ & x_1, x_2, x_3 \geq 0 \end{array}$$

1. For any feasible solution: $z = 4x_1 + x_2 + 3x_3 \leq 11x_1 + 5x_2 + 3x_3 \leq 11$,
2. Opt solution is LP L is somewhere between 9 and 11.
3. Multiply first inequality by y_1, second inequality by y_2 and add them up:

$$\begin{array}{rcccccc} y_1(x_1 & + & 4x_2 & & &) \leq & y_1(1) \\ +\ y_2(3x_1 & - & x_2 & + & x_3 &) \leq & y_2(3) \\ \hline (y_1 + 3y_2)x_1 & + & (4y_1 - y_2)x_2 & + & y_2x_3 & \leq & y_1 + 3y_2. \end{array}$$

Duality by Example: IV

$$
\begin{array}{lrl}
\max & z = & 4x_1 + x_2 + 3x_3 \\
\text{s.t.} & & x_1 + 4x_2 \le 1 \\
& & 3x_1 - x_2 + x_3 \le 3 \\
& & x_1, x_2, x_3 \ge 0
\end{array}
$$

1. $(y_1 + 3y_2)x_1 + (4y_1 - y_2)x_2 + y_2 x_3 \le y_1 + 3y_2$.

$$
\begin{array}{rcl}
4 & \le & y_1 + 3y_2 \\
1 & \le & 4y_1 - y_2 \\
3 & \le & y_2,
\end{array}
$$

1. Compare to target function – require expression bigger than target function in each variable.

$\implies z = 4x_1 + x_2 + 3x_3 \le$
$(y_1 + 3y_2)x_1 + (4y_1 - y_2)x_2 + y_2 x_3 \le y_1 + 3y_2$.

Duality by Example: IV

Primal LP:

$$
\begin{array}{lrl}
\max & z = & 4x_1 + x_2 + 3x_3 \\
\text{s.t.} & & x_1 + 4x_2 \le 1 \\
& & 3x_1 - x_2 + x_3 \le 3 \\
& & x_1, x_2, x_3 \ge 0
\end{array}
$$

Dual LP: $\widehat{L}$

$$
\begin{array}{ll}
\min & y_1 + 3y_2 \\
\text{s.t.} & y_1 + 3y_2 \ge 4 \\
& 4y_1 - y_2 \ge 1 \\
& y_2 \ge 3 \\
& y_1, y_2 \ge 0.
\end{array}
$$

1. Best upper bound on η (max value of z) then solve the LP $\widehat{L}$.
2. $\widehat{L}$: Dual program to L.
3. opt. solution of $\widehat{L}$ is an upper bound on optimal solution for L.

Primal program/Dual program

$$
\begin{array}{ll}
\max & \sum_{j=1}^{n} c_j x_j \\
\text{s.t.} & \sum_{j=1}^{n} a_{ij} x_j \le b_i, \\
& \qquad \text{for } i = 1, \ldots, m, \\
& x_j \ge 0, \\
& \qquad \text{for } j = 1, \ldots, n.
\end{array}
$$

$$
\begin{array}{ll}
\min & \sum_{i=1}^{m} b_i y_i \\
\text{s.t.} & \sum_{i=1}^{m} a_{ij} y_i \ge c_j, \\
& \qquad \text{for } j = 1, \ldots, n, \\
& y_i \ge 0, \\
& \qquad \text{for } i = 1, \ldots, m.
\end{array}
$$

Primal program/Dual program

Dual variables \ *Primal variables*	$x_1 \geqq 0$	$x_2 \geqq 0$	$x_3 \geqq 0$	$\cdots$	$x_n \geqq 0$	*Primal relation*	Min v
$y_1 \geqq 0$	a_{11}	a_{12}	a_{13}	$\cdots$	a_{1n}	$\leqq$	b_1
$y_2 \geqq 0$	a_{21}	a_{22}	a_{23}	$\cdots$	a_{2n}	$\leqq$	b_2
$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$
$y_m \geqq 0$	a_{m1}	a_{m2}	a_{m3}	$\cdots$	a_{mn}	$\leqq$	b_m
Dual Relation	$\geqq$	$\geqq$	$\geqq$		$\geqq$		
Max z	c_1	c_2	c_3	$\cdots$	c_n		

$$
\begin{array}{ll}
\max & c^T x \\
\text{s. t.} & Ax \le b. \\
& x \ge 0.
\end{array}
\qquad
\begin{array}{ll}
\min & y^T b \\
\text{s. t.} & y^T A \ge c^T. \\
& y \ge 0.
\end{array}
$$

Primal program/Dual program

What happens when you take the dual of the dual?

$$\begin{aligned} \max \quad & \sum_{j=1}^{n} c_j x_j \\ \text{s.t.} \quad & \sum_{j=1}^{n} a_{ij} x_j \leq b_i, \\ & \qquad \text{for } i = 1, \ldots, m, \\ & x_j \geq 0, \\ & \qquad \text{for } j = 1, \ldots, n. \end{aligned}$$

$$\begin{aligned} \min \quad & \sum_{i=1}^{m} b_i y_i \\ \text{s.t.} \quad & \sum_{i=1}^{m} a_{ij} y_i \geq c_j, \\ & \qquad \text{for } j = 1, \ldots, n, \\ & y_i \geq 0, \\ & \qquad \text{for } i = 1, \ldots, m. \end{aligned}$$

Primal program / Dual program in standard form

$$\begin{aligned} \max \quad & \sum_{j=1}^{n} c_j x_j \\ \text{s.t.} \quad & \sum_{j=1}^{n} a_{ij} x_j \leq b_i, \\ & \qquad \text{for } i = 1, \ldots, m, \\ & x_j \geq 0, \\ & \qquad \text{for } j = 1, \ldots, n. \end{aligned}$$

$$\begin{aligned} \max \quad & \sum_{i=1}^{m} (-b_i) y_i \\ \text{s.t.} \quad & \sum_{i=1}^{m} (-a_{ij}) y_i \leq -c_j, \\ & \qquad \text{for } j = 1, \ldots, n, \\ & y_i \geq 0, \\ & \qquad \text{for} i = 1, \ldots, m. \end{aligned}$$

Dual program in standard form / Dual of dual program

$$\begin{aligned} \max \quad & \sum_{i=1}^{m} (-b_i) y_i \\ \text{s.t.} \quad & \sum_{i=1}^{m} (-a_{ij}) y_i \leq -c_j, \\ & \qquad \text{for } j = 1, \ldots, n, \\ & y_i \geq 0, \\ & \qquad \text{for } i = 1, \ldots, m. \end{aligned}$$

$$\begin{aligned} \min \quad & \sum_{j=1}^{n} -c_j x_j \\ \text{s.t.} \quad & \sum_{j=1}^{n} (-a_{ij}) x_j \geq -b_i, \\ & \qquad \text{for } i = 1, \ldots, m, \\ & x_j \geq 0, \\ & \qquad \text{for } j = 1, \ldots, n. \end{aligned}$$

Dual of dual program / Dual of dual program written in standard form

$$\begin{aligned} \min \quad & \sum_{j=1}^{n} -c_j x_j \\ \text{s.t.} \quad & \sum_{j=1}^{n} (-a_{ij}) x_j \geq -b_i, \\ & \qquad \text{for } i = 1, \ldots, m, \\ & x_j \geq 0, \\ & \qquad \text{for } j = 1, \ldots, n. \end{aligned}$$

$$\begin{aligned} \max \quad & \sum_{j=1}^{n} c_j x_j \\ \text{s.t.} \quad & \sum_{j=1}^{n} a_{ij} x_j \leq b_i, \\ & \qquad \text{for } i = 1, \ldots, m, \\ & x_j \geq 0, \\ & \qquad \text{for } j = 1, \ldots, n. \end{aligned}$$

$\Longrightarrow$ Dual of the dual LP is the primal LP!

Result

Proved the following:

Lemma

Let $\boldsymbol{L}$ be an LP, and let $\boldsymbol{L'}$ be its dual. Let $\boldsymbol{L''}$ be the dual to $\boldsymbol{L'}$. Then $\boldsymbol{L}$ and $\boldsymbol{L''}$ are the same LP.

Weak duality theorem

Theorem

If $(x_1, x_2, \ldots, x_n)$ is feasible for the primal LP and $(y_1, y_2, \ldots, y_m)$ is feasible for the dual LP, then

$$\sum_j c_j x_j \le \sum_i b_i y_i.$$

Namely, all the feasible solutions of the dual bound all the feasible solutions of the primal.

Weak duality theorem – proof

Proof.

By substitution from the dual form, and since the two solutions are feasible, we know that

$$\sum_j c_j x_j \le \sum_j \left(\sum_{i=1}^m y_i a_{ij} \right) x_j \le \sum_i \left(\sum_j a_{ij} x_j \right) y_i \le \sum_i b_i y_i \; .$$

□

1. $\boldsymbol{y}$ being dual feasible implies $\boldsymbol{c}^T \le \boldsymbol{y}^T \boldsymbol{A}$
2. $\boldsymbol{x}$ being primal feasible implies $\boldsymbol{A}\boldsymbol{x} \le \boldsymbol{b}$
3. $\Rightarrow \boldsymbol{c}^T \boldsymbol{x} \le (\boldsymbol{y}^T \boldsymbol{A})\boldsymbol{x} \le \boldsymbol{y}^T(\boldsymbol{A}\boldsymbol{x}) \le \boldsymbol{y}^T \boldsymbol{b}$

The strong duality theorem

Theorem (Strong duality theorem.)

If the primal LP problem has an optimal solution $x^* = \left(x_1^*, \ldots, x_n^*\right)$ then the dual also has an optimal solution, $y^* = \left(y_1^*, \ldots, y_m^*\right)$, such that

$$\sum_j c_j x_j^* = \sum_i b_i y_i^*.$$