Existential Quantification

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Algorithms and Complexity

Let $R \subseteq \{0,1\}^*$ be decidable. Consider

„$\exists R$“ $:= \{\, \underline{x} \in \{0,1\}^* \mid \exists \underline{y} \in \{0,1\}^* : \langle \underline{x}, \underline{y} \rangle \in R \,\}$

- semidecidable; conversely, semidecidable
- $L = \{\, \underline{x} \in \{0,1\}^* \mid \exists n$: program P, on input $\underline{x}$, terminates after n steps $\}$
- $R := \{\, \langle P, \underline{x}, n \rangle \mid$ program P terminates after n steps on input $\underline{x}\,\}$ decidable (UTM)

{ „$\exists R$“ : R decidable } =
{ semi-decidable/enumerable problems }

Digression: class **NP**

For $R \subseteq \{0,1\}^*$ decidable, $\{ \underline{x} \in \{0,1\}^* \mid \exists \underline{y} \in \{0,1\}^* : \langle \underline{x}, \underline{y} \rangle \in R \}$ semidecidable; any semidecidable problem has this form

For $R \subseteq \{0,1\}^*$ decidable in polynomial time (**P**),

$$\{ \underline{x} \mid \exists \underline{y} \in \{0,1\}^{p(|\underline{x}|)} : \langle \underline{x}, \underline{y} \rangle \in R \}$$

decidable in nondeterministic polynomial time (**NP**).

Any problem in **NP** has this form.

Example: $\{ \langle G \rangle \mid \exists C$: C is Hamilton Cycle in $G \}$

$1,000,000: **P** = **NP** ?

Alternating Quantifiers

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$$\text{“}\exists R\text{“} := \{\, \underline{x}\in\{0,1\}^* \mid \exists \underline{y}\in\{0,1\}^* : \langle \underline{x},\underline{y}\rangle \in R \,\}$$

$$\text{“}\forall R\text{“} := \{\, \underline{x}\in\{0,1\}^* \mid \forall \underline{y}\in\{0,1\}^* : \langle \underline{x},\underline{y}\rangle \in R \,\}$$

- $\Sigma_0 := \Pi_0 := \Sigma_1 \cap \Pi_1$ decidable problems
- $\Sigma_1 := \{$ “$\exists R$“ : R decidable $\}$
  = { semi-decidable problems }
- $\Pi_1 := \{$ “$\forall R$“ : R decidable $\}$
  = { complements of semi-decidable problems }
- $\Sigma_2 := \{$ “$\exists\forall R$“ : R decidable $\}$ = $\{$ “$\exists L$“ : $L \in \Pi_1 \}$
- $\Pi_2 := \{$ “$\forall\exists R$“ : R decidable $\}$ = $\{$ “$\forall L$“ : $L \in \Sigma_1 \}$

$$\{\, \underline{x}\in\{0,1\}^* \mid \exists \underline{y}\in\{0,1\}^* \ \forall \underline{z}\in\{0,1\}^* : \langle \underline{x},\underline{y},\underline{z}\rangle \in R \,\}$$

Go Figure (I)

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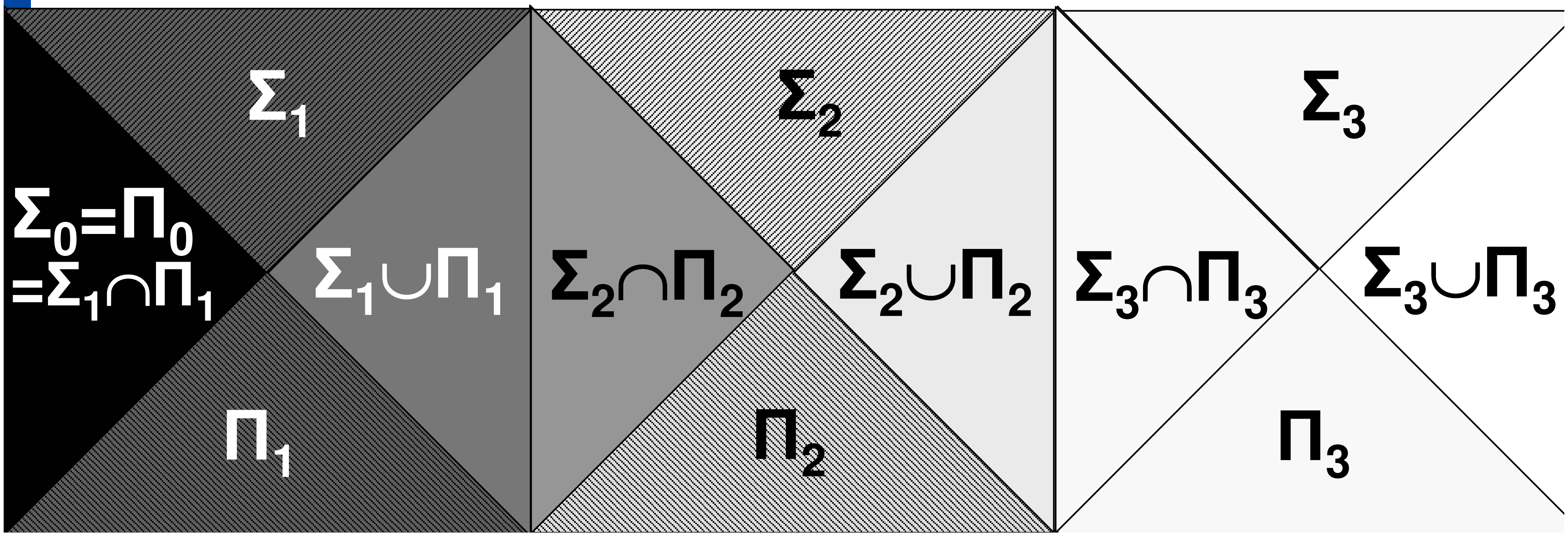
$\Sigma_d \;=\; \{\, \text{“}\exists\forall\exists \ldots \;\; R\text{“} \mid R \text{ decidable} \,\}$

($d\in\mathbb{N}$ alternating quantifiers)

$\Pi_d \;=\; \{\, \text{“}\forall\exists\forall \ldots \;\; R\text{“} \mid R \text{ decidable} \,\}$



Syntactical versus Semantical
Arithmetical Hierarchy

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- Σ_0 decidable problems
- Σ_1 = { "$\exists R$" : R decidable }
  = { semi-decidable problems }
- Σ_2 = { "$\exists\forall R$" : R decidable }

Theorem: For $L \subseteq \{0,1\}^*$ it holds:

a) L semi-decidable relative to H $\Leftrightarrow$ $L \in \Sigma_2$

b) L decidable relative to H $\Leftrightarrow$ $L \in \Sigma_2 \cap \Pi_2$

c) L semi-decidable relative to $\emptyset^{(d)}$ $\Leftrightarrow$ $L \in \Sigma_{d+1}$

Corollary: $H^H \in \Sigma_2 \setminus \Pi_2$, $\emptyset^{(d)} \in \Sigma_d \setminus \Pi_d$

Go Figure (II)

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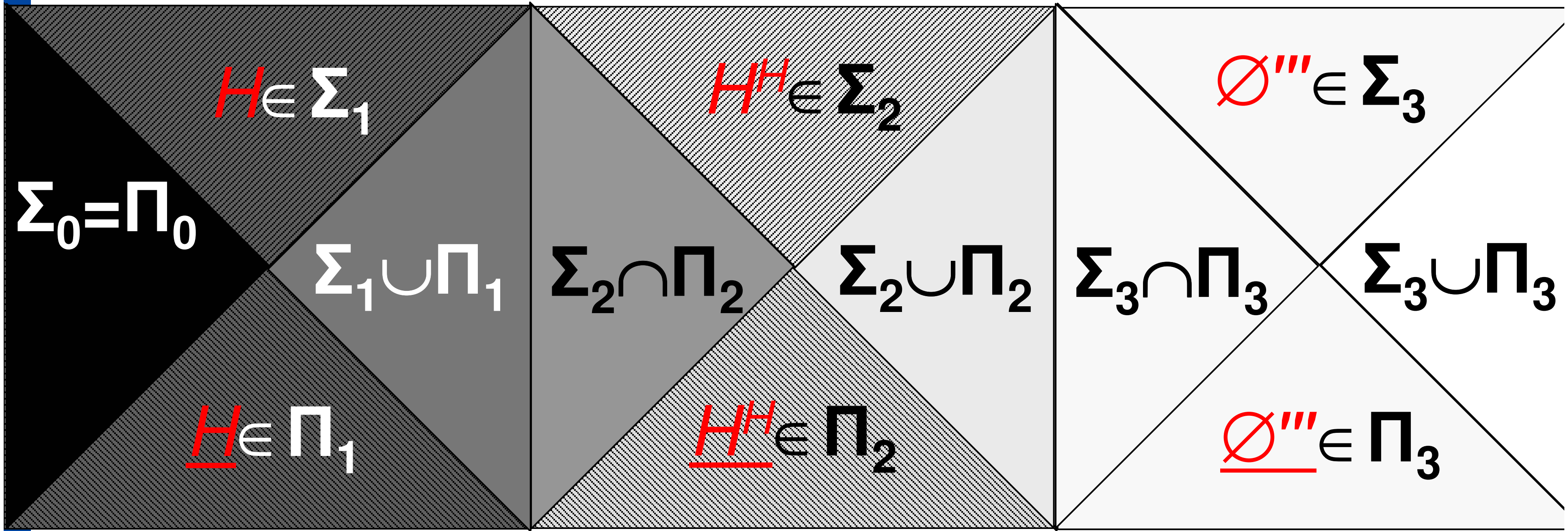
complements: $\underline{H^H} \in \Pi_2 \setminus \Sigma_2$, of $\underline{\emptyset^{(d)}} \in \Pi_{d+1} \setminus \Sigma_d$



Corollary: $H^H \in \Sigma_2 \setminus \Pi_2$, $\emptyset^{(d)} \in \Sigma_d \setminus \Pi_d$