Geometric Permutations of Disjoint Unit Spheres[1]

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Abstract

We show that a set of n disjoint unit spheres in $\mathbb{R}^d$ admits at most two distinct geometric permutations if $n \geq 9$, and at most three if $3 \leq n \leq 8$. This result improves a Helly-type theorem on line transversals for disjoint unit spheres in $\mathbb{R}^3$: if any subset of size 18 of a family of such spheres admits a line transversal, then there is a line transversal for the entire family.

1 Introduction

A *line transversal* for a set $\mathcal{F}$ of pairwise disjoint convex bodies in $\mathbb{R}^d$ is a line ℓ that intersects every element of $\mathcal{F}$. A line transversal induces two linear orders on $\mathcal{F}$, namely the orders in which the two possible orientations of ℓ intersect the elements of $\mathcal{F}$. Since the two orders are the reverse of each other, we consider them as a single *geometric permutation*.

Bounds on the maximum number of geometric permutations were established about a decade ago: a tight bound of $2n-2$ is known for two dimensions [3], for higher dimension the number is in $\Omega(n^{d-1})$ [8] and in $O(n^{2d-2})$ [13]. The gap was closed for the special case of spheres by Smorodinsky et al. [12], who showed that n spheres in $\mathbb{R}^d$ admit $\Theta(n^{d-1})$ geometric permutations. This result can be generalized to "fat" convex objects [10].

The even more specialized case of congruent spheres was treated by Smorodinsky et al. [12] and independently by Asinowski [1]. They proved that n unit circles in $\mathbb{R}^2$ admit at most two geometric permutations if n is large enough (the proof by Asinowski holds for all $n \geq 4$). Zhou and Suri established an upper bound of 16 for all d, if n is sufficiently large, a result quickly improved by Katchalski, Suri, and Zhou [9] and independently by Huang, Xu, and Chen [6] to 4.

Building on Katchalski et al.'s proof, we recently showed that there are in fact at most two geometric permutations [2]. As two geometric permutations are possible for any n, this bound is optimal. However, Katchalski et al.'s approach—and therefore our extension to it as well—relies strongly on the assumption that n is "sufficiently" large, which implies that any two line transversals of $\mathcal{F}$ are nearly parallel. The critical threshold has been estimated to be about 31 in 3 dimensions [5], but it increases exponentially with d. The proof gives no bound on the number of geometric permutations of n spheres if n is smaller than this threshold.

In the present paper we analyze line transversals for unit spheres in $\mathbb{R}^d$ in more detail. In particular, we prove that n disjoint unit spheres admit at most three geometric permutations, for any n, and at most two geometric permutations for $n \geq 9$.

We prove these bounds by showing that some pairs of geometric permutations are *incompatible*. Let $\mathcal{F}$ be a family of disjoint convex objects (not necessarily spheres) in $\mathbb{R}^d$. A pair of geometric permutations, such as $(ABCD, BADC)$, is *incompatible* if no set of four objects

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$A, B, C, D \in \mathcal{F}$ admits both a line transversal realizing $ABCD$ and a line transversal realizing $BADC$.

Our first result is that if the pairs $(ABCD, BADC)$ and $(ABCD, ADCB)$ are both incompatible for a family $\mathcal{F}$, then $\mathcal{F}$ admits at most 3 geometric permutations. This fact was, in a sense, already used by Katchalski et al. [7, 8], but proven only for translates in the plane. In fact, the result can be proven purely combinatorially. We then show that if the two additional pairs $(ABCD, ADBC)$ and $(ABCD, CADB)$ are incompatible as well, then $\mathcal{F}$ admits at most two geometric permutations that differ by the swapping of a single pair of adjacent objects.

To prove the incompatibility of $(ABCD, ADCB)$, we show that a line transversal that meets three unit spheres S, U, and T in that order makes an angle of less than $45°$ with the line through the centers of S and T. This bound is tight, and settles a problem posed by Holmsen et al. [5], who had conjectured the angle to be at most $60°$.

Next, and maybe the cornerstone of this paper, we prove that the pair $(ABCD, BADC)$ is incompatible for disjoint unit spheres. This is nearly trivial in the plane, even for arbitrary convex objects, but takes considerable effort to prove for unit spheres in higher dimensions. The claim does not hold for general convex sets here, not even for spheres of different radia, or for unit spheres that are allowed to overlap somewhat. The bound of three geometric permutations for any family of disjoint unit spheres in any dimension follows.

We then establish that the pairs $(ABCD, ADBC)$ and $(ABCD, CADB)$ can be compatible only if the two line transversals make an angle of at least $45°$ with each other. We show that it is impossible for any set of 9 unit spheres to admit two line transversals with such a large angle, and thus obtain the bound of two geometric permutations for at least 9 unit spheres, with the two permutations differing only by the swapping of two adjacent spheres. We conjecture that the two pairs in question are in fact incompatible, which would imply the bound for any $n > 3$.

Surveys of geometric transversal theory are Goodman et al. [4] and Wenger [14]. The latter also discusses Helly-type theorems for line transversals. A recent result in that area by Holmsen et al. [5] proves the existance of a number n_0 such that the following holds: Let $\mathcal{F}$ be a set of disjoint unit spheres in $\mathbb{R}^3$. If every n_0 members of $\mathcal{F}$ have a line transversal, then $\mathcal{F}$ has a line transversal. Holmsen et al.'s proof implies $n_0 \leq 46$. Our results imply $n_0 \leq 18$.

2 Incompatible pairs and geometric permutations

In this section we show that the incompatibility of certain pairs of geometric permutations implies a bound on the number of geometric permutations. The proofs are purely combinatorial, and apply to any family $\mathcal{F}$ of convex disjoint objects in $\mathbb{R}^d$.

Lemma 1 *If a set of $n \geq 4$ objects of $\mathcal{F}$ admits three distinct geometric permutations, then there are 4 objects in $\mathcal{F}$ that admit three distinct geometric permutations.*

Proof. Let t_1, t_2 and t_3 be three geometric permutations of the family $\mathcal{F}$. Let t_1 begin with AB, and t_2 and t_3 be oriented so that A appears before B.

First, we can assume that t_2 does not begin with AB. Indeed, if both t_2 and t_3 begin with AB, then t_1, t_2 and t_3 realize 3 distinct geometric permutations of $\mathcal{F} \setminus \{A\}$. If $n \leq 5$, then the proof is complete because $\mathcal{F} \setminus \{A\}$ is a set of 4 objects having 3 distinct geometric permutations. If $n \geq 6$, we restart our reasoning with the set $\mathcal{F} \setminus \{A\}$.

Next, let C be one of the two first objects stabbed by t_2 and distinct from A and B. If the restriction of t_3 to the objects A, B, C is distinct from those of t_1 and t_2, then we can add any fourth object of $\mathcal{F}$ to $\{A, B, C\}$, and the t_i realize three distinct geometric permutations of those four objects. If the restriction of t_3 is equal to that of t_j, for $j = 1$ or 2, then there must be some pair of objects (D, E) that are met in different orders by t_j and t_3. The objects $\{A, B, C, D, E\}$ admit three distinct geometric permutations.

If t_j and t_3 meet one of the sets A, B, C, D and A, B, C, E in distinct orders, then those four objects have three distinct geometric permutations, and the result is proven. Otherwise, the restrictions of t_j and t_3 to A, B, C, D, E only differ in a switch of D and E.

In t_1, the objects D and E appear after object B. If one of them, say D, appears before object B in t_2, then the t_i realize three distinct geometric permutations of A, B, D, E: t_1 and t_2 differ in the fact that B and D are switched, and t_3 differs from the others in the switching of D and E.

The last remaining case is when the objects D and E appear after object B in each of the t_i. In that case, the t_i realise three distinct geometric permutations of B, C, D, E: t_1 and t_2 differ in the fact that B and C are switched, and t_3 differ from one of the others in the switching of D and E. □

Lemma 2 *Let $\mathcal{F}$ be a family of n disjoint convex objects such that the pairs $(ABCD, BADC)$ and $(ABCD, ADCB)$ are incompatible. Then $\mathcal{F}$ admits at most three geometric permutations.*

Proof. Assume $\mathcal{F}$ admits four different geometric permutations. By Lemma 1 there is a subset of four objects $A, B, C, D \in \mathcal{F}$ that admits three different geometric permutations. The following table shows the 12 different geometric permutations of these four objects:

A B C D	A B D C	A C B D
A D C B	D C A B	A D B C
D A B C	A C D B	D A C B
C D A B	C A B D	C A D B

Here, any pair of permutations from the same column form an incompatible pair. On the other hand, any triple of permutations taken from all three columns contains a subset A, B, C of three objects that appears in all three possible geometric permutations.

Consider now the fourth geometric permutation of $\mathcal{F}$. Its restriction to A, B, C must coincide with one of the first three permutations, and so there must be two objects D and E (one of which could be in the set $\{A, B, C\}$) that distinguish the fourth permutation. This implies that we have identified at most five objects $\{A, B, C, D, E\}$ that admit four geometric permutations.

Perhaps the easiest way to complete the proof is to check the resulting finite number of cases mechanically.[1] An elementary proof using nothing but case distinctions is possible, but lengthy and not illuminating. □

Lemma 3 *Let $\mathcal{F}$ be a family of disjoint convex objects with $(ABCD, BADC)$, $(ABCD, ADCB)$, $(ABCD, ADBC)$, and $(ABCD, CADB)$ as incompatible pairs. Then $\mathcal{F}$ has at most two distinct geometric permutations that differ only in the swapping of a single pair of adjacent objects*

Proof. Let ℓ and ℓ' be two line transversals for $\mathcal{F}$ realizing distinct geometric permutations. We first prove the following *claim (i)*: If two objects A and D appear in consecutive positions in the geometric permutation realized by ℓ, then at most one other object can appear in between A and D in the geometric permutation realized by ℓ'. Indeed, assume A and D appear separated by two other objects in ℓ', so that ℓ' realizes $ABCD$. If B and C appear on opposite sides of the pair AD in ℓ, then ℓ realizes either $BADC$ or $CADB$, a contradiction. If B and C appear on one side, we can assume (by renaming the objects) that ℓ realizes either $ADBC$ or $ADCB$, a contradiction.

We now number the objects in the order in which they are intersected by ℓ, and denote them $B_1, B_2, \ldots, B_n$. Let similarly $B'_1, B'_2, \ldots, B'_n$ be the order in which they are intersected by ℓ'.

[1]To double-check our lengthy proof we have indeed written a small program that generates all sets of 4 distinct geometric permutations of 5 objects and verifies that it contains an incompatible pair.

We prove the following *claim (ii)*: If, for some i, we have $\{B'_1, \dots, B'_i\} = \{B_1, \dots, B_i\}$ and $B'_i = B_i$, then either $B'_{i+1} = B_{i+1}$, or $B'_{i+1} = B_{i+2}$, $B'_{i+2} = B_{i+1}$, and $B'_{i+3} = B_{i+3}$. Indeed, if $B'_{i+1} = B_j$ with $j > i+2$, then B_i and B_j are adjacent in ℓ', but separated by B_{i+1} and B_{i+2} in ℓ, a contradiction by claim (i). If $B'_{i+1} = B_{i+1}$, we have the first case of the claim, so it rests to consider $B'_{i+1} = B_{i+2}$. Then B'_{i+2} must be B_{i+1} (otherwise, B_i and B_{i+1} are adjacent in ℓ but separated by two objects in ℓ'), and finally $B'_{i+3} = B_{i+3}$ (otherwise B_{i+2} and B_{i+3} are adjacent in ℓ, but separated by two objects in ℓ').

If $B'_1 = B_1$, we can repeatedly apply claim (ii) to observe that ℓ and ℓ' can differ only by the exchange of independent adjacent pairs. There cannot be more than one such pair since $(ABCD, BADC)$ is incompatible, and so the lemma follows.

It remains to consider the case $B'_1 \neq B_1$. Let $B'_j = B_1$, with $1 < j < n$ (if $B'_n = B_1$ we reverse the numbering of objects along ℓ' and apply the previous argument). We observe that then $\{B'_{j-1}, B'_{j+1}\} = \{B_2, B_3\}$ since no other object can appear adjacent to B_1 in ℓ'. Without loss of generality, let $B'_{j-1} = B_2$, $B'_{j+1} = B_3$ (otherwise we reverse the numbering of objects along ℓ'). Now, B_4 cannot appear before B'_{j-1} (that is, as $B'_1, \dots, B'_{j-2}$), and inductively it follows that *no* object can appear before B'_{j-1}. This implies $j = 2$, and we have $\{B'_1, B'_2, B'_3\} = \{B_1, B_2, B_3\}$ with $B'_3 = B_3$. Once again we can kickstart claim (ii) to prove the lemma. □

3 Unit spheres and their transversals

A *unit sphere* is a sphere of radius 1. We say that two unit spheres are *disjoint* if their interiors are (in other words, we allow the spheres to touch). A line *stabs* a sphere if it intersects the closed sphere (and so a tangent to a sphere stabs it). A *line transversal* for a set of disjoint unit spheres is a line that stabs all the spheres, with the restriction that it is not allowed to be tangent to two spheres in a common point (as such a line does not define a geometric permutation).

We will denote unit spheres by upper-case letters $A, B, \dots$, and use the corresponding lower-case letters $a, b, \dots$ for their centers. We make no distinction between points and vectors, so the vector from the center of sphere A to the center of sphere B is $b - a$.

Given two disjoint unit spheres A and B, let $\Pi(A, B)$ be their bisecting hyperplane. In other words, $\Pi(A, B)$ is the hyperplane through $(a + b)/2$ with normal $b - a$. We use $d(\cdot, \cdot)$ to denote the Euclidean distance of two points, that is $d(a, b)^2 = (b - a)^2$.

Our discussion will make heavy use of angles between lines and subspaces, so it is worthwhile clarifying their definition. Let $u \cdot v$ denote the dot-product of two vectors u and v. The angle between two vectors u and v is $\arccos \frac{u \cdot v}{\|u\|\|v\|}$. Consider now a k-dimensional flat (affine subspace) Γ, and a line ℓ intersecting Γ. Let u be the direction vector of ℓ, that is, we can write ℓ as $\{p + \lambda u \mid \lambda \in \mathbb{R}\}$. We can express u uniquely as $u = v + w$, where v is contained in Γ, and $v \cdot w = 0$. The angle between ℓ and Γ is defined as the angle between the vectors u and v. The angle is zero if and only if ℓ is contained in Γ. Note that the angle does not depend on the orientation chosen for the line, and does not change if the line is replaced by a parallel line, or the flat by a parallel flat.

We start with a warm-up lemma in two dimensions.

Lemma 4 *Let S and T be two unit-radius disks in $\mathbb{R}^2$ with centers $(-\lambda, 0)$ and $(\lambda, 0)$, where $\lambda \geq \cos\beta$ for some angle β with $0 < \beta \leq \pi/2$. Then $S \cap T$ is contained in the ellipse*

$$\left(\frac{x}{\sin^2\beta}\right)^2 + \left(\frac{y}{\sin\beta}\right)^2 \leq 1.$$

Proof. Let $(\mu, 0)$ and $(0, \nu)$ be the rightmost and topmost point of $S \cap T$ (see Figure 1). Consider

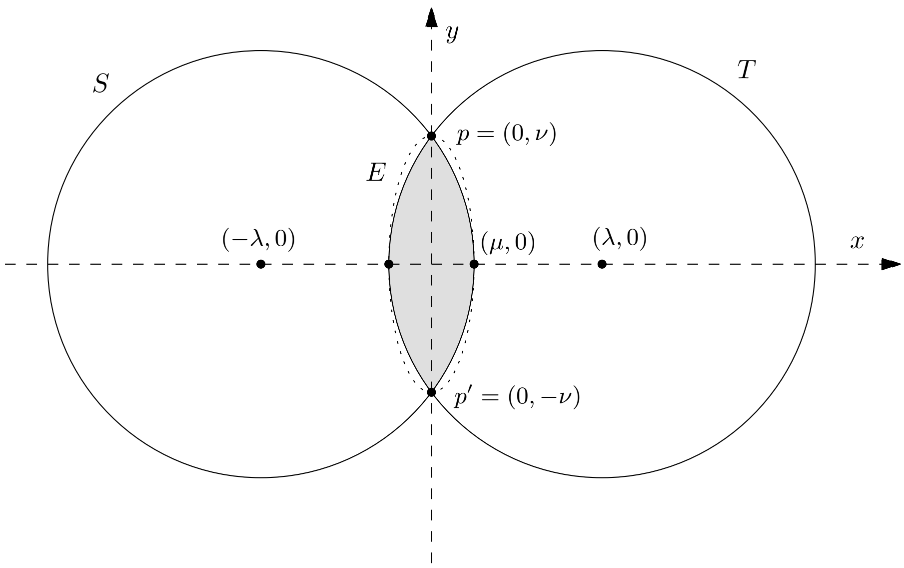

Figure 1: The intersection of two disks is contained in an ellipse.

the ellipse E defined as

$$(\frac{x}{\mu})^2 + (\frac{y}{\nu})^2 \leq 1.$$

E intersects the boundary of S in $p = (0, \nu)$ and $p' = (0, -\nu)$, and is tangent to it in $(\mu, 0)$. An ellipse can intersect a circle in at most four points and the tangency counts as two intersections, and so the intersections at p and p' are proper and there is no further intersection between the two curves. This implies that the boundary of E is divided into two pieces by p and p', with one piece inside S and one outside S. Since $(-\mu, 0)$ lies inside S, the right hand side of E lies outside S. Symmetrically, the left hand side of E lies outside T, and so $S \cap T$ is contained in E. It remains to observe that

$$\nu^2 = 1 - \lambda^2 \leq 1 - \cos^2 \beta = \sin^2 \beta,$$

so $\nu \leq \sin \beta$, and

$$\mu = 1 - \lambda \leq 1 - \cos \beta \leq 1 - \cos^2 \beta = \sin^2 \beta,$$

which proves the lemma. □

We now show that a transversal for two spheres cannot pass too far from their common center of gravity.

Lemma 5 *Given two disjoint unit spheres* A *and* B *in* $\mathbb{R}^d$ *and a line* ℓ *stabbing both spheres, let* p *be the point of intersection of* ℓ *and* $\Pi(A, B)$*, and let* β *be the angle between* ℓ *and* $\Pi(A, B)$*. Then*

$$d(p, (a + b)/2) \leq \sin \beta.$$

Proof. Let a and b be the centers of A and B and let ν be the direction vector of ℓ, that is, ℓ can be written as $\{p + \lambda\nu \mid \lambda \in \mathbb{R}\}$. We first argue that proving the lemma for $d = 3$ is sufficient. Indeed, assume $d > 3$ and consider the 3-dimensional subspace Γ containing ℓ, a, and b. Since we have $d(a, \ell) \leq 1$ and $d(b, \ell) \leq 1$, the line ℓ stabs the 3-dimensional unit spheres $A \cap \Gamma$ and $B \cap \Gamma$. And since $\pi/2 - \beta$ is the angle between two vectors in Γ, namely ν and $b - a$, β is also the angle between ℓ and the two-dimensional plane $\Pi(A, B) \cap \Gamma$. So if the lemma holds in Γ, then it also holds in $\mathbb{R}^d$.

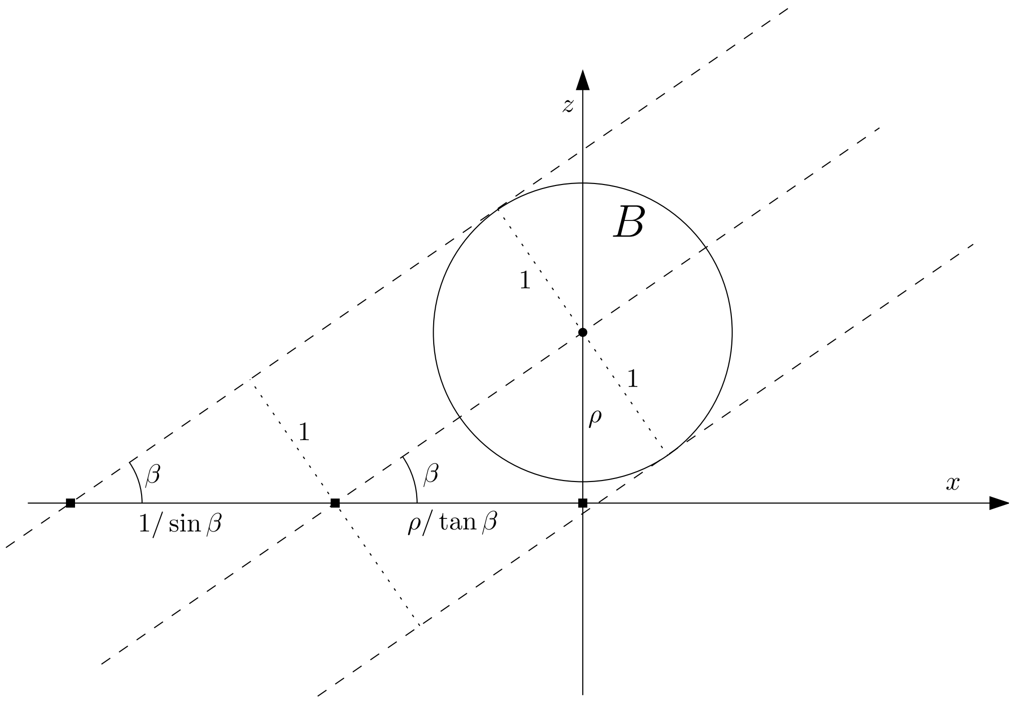

Figure 2: The intersection of the cylinder with the xy-plane is an ellipse.

In the rest of the proof we can therefore assume that $d = 3$. We choose a coordinate system where $a = (0, 0, -\rho)$, $b = (0, 0, \rho)$ with $\rho \geq 1$, and $v = (\cos\beta, 0, \sin\beta)$. Then $\Pi := \Pi(A, B)$ is the xy-plane and $g := (a + b)/2 = (0, 0, 0)$. Consider the cylinders $\mathrm{cyl}(A) := \{u + \lambda v \mid u \in A, \lambda \in \mathbb{R}\}$ and $\mathrm{cyl}(B)$ defined accordingly. Since ℓ stabs A and B, we have $p \in \mathrm{cyl}(A) \cap \mathrm{cyl}(B) \cap \Pi$.

The intersection $B' := \mathrm{cyl}(B) \cap \Pi$ is the ellipse (see Figure 2)

$$\sin^2\beta(x + \frac{\rho}{\tan\beta})^2 + y^2 \leq 1,$$

and symmetrically $A' := \mathrm{cyl}(A) \cap \Pi$ is

$$\sin^2\beta(x - \frac{\rho}{\tan\beta})^2 + y^2 \leq 1.$$

If we let τ be the linear transformation

$$\tau : (x, y) \mapsto (x \sin\beta, y),$$

then $\tau(A')$ and $\tau(B')$ are unit-radius disks with centers $(\rho\cos\beta, 0)$ and $(-\rho\cos\beta, 0)$. By Lemma 4, the intersection $\tau(A' \cap B')$ is contained in the ellipse

$$\left(\frac{x}{\sin^2\beta}\right)^2 + \left(\frac{y}{\sin\beta}\right)^2 \leq 1.$$

Applying τ^{-1} we find that $A' \cap B'$ is contained in the circle with radius $\sin\beta$ around g. Since $p \in A' \cap B'$, the lemma follows. ◻

Let ℓ be a line transversal for a family $\mathcal{S}$ of n disjoint unit spheres in $\mathbb{R}^d$. This implies that the center of any sphere in $\mathcal{S}$ lies inside a cylinder of radius 1 around ℓ. A volume argument [9] shows that the distance between the first and the last sphere is $\Omega(n)$, with a constant depending exponentially on the dimension d. The following lemma improves this to the absolute constant $\sqrt{2}$, which is easily seen to be tight in any dimension.

Lemma 6 *Let $\mathcal{C}$ be a cylinder of radius 1 and length less than $s\sqrt{2}$, for some $s \in \mathbb{N}$. Then $\mathcal{C}$ contains at most $2s$ points with pairwise distance at least 2.*

Proof. Let the axis of $\mathcal{C}$ be the x_1-axis, assume $\mathcal{C}$ contains at least $2s+1$ points, and partition it into s pieces of length less than $\sqrt{2}$. One of these pieces must contain at least three points a, b, c. We can assume $0 = a_1 \leq b_1 \leq c_1 < \sqrt{2}$. We increase c_1 to $\sqrt{2}$—this will increase $d(a, c)$ and $d(b, c)$ so that we have $d(b, c) > 2$. Let a', b', c' be the projection of the points on the hyperplane $x_1 = 0$. These points are contained in a unit sphere S with center in the origin. Let Π be the two-dimensional plane containing a', b', c'. It intersects S in a disk of radius at most 1. Let p be the center of this disk. The pairwise distance of the points a', b', c' is at least $\sqrt{2}$, as the pairwise difference of a_1, b_1, c_1 is at most $\sqrt{2}$. It follows that the angles $\angle a'pb', \angle b'pc', \angle c'pa'$ are all at least $\pi/2$. This implies that moving all three points away from p can only increase their pairwise distances, and so we can assume $d(p, a') = d(p, b') = d(p, c') = 1$. Furthermore, we can rotate c' around p towards a' until $\angle a'pc' = \pi/2$, as this can only increase $d(b', c')$. We have

$$\begin{aligned} 4 &\leq d(a,b)^2 = d(a',b')^2 + b_1^2, \\ 4 &< d(b,c)^2 = d(b',c')^2 + (\sqrt{2} - b_1)^2, \end{aligned}$$

Let now $a'' = p + (p - a')$ and $c'' = p + (p - c')$. The point b' lies somewhere on the quarter circle around p between a'' and c''. By Thales' theorem, the angles $\angle a''b'a'$ and $\angle c''b'c'$ are right angles, so we have

$$\begin{aligned} d(b', a'')^2 &= d(a', a'')^2 - d(a', b')^2 = 4 - d(a', b')^2 \leq b_1^2, \\ d(b', c'')^2 &= d(c', c'')^2 - d(c', b')^2 = 4 - d(c', b')^2 < (\sqrt{2} - b_1)^2. \end{aligned}$$

This implies $d(b', a'') \leq b_1$ and $d(b', c'') < \sqrt{2} - b_1$. By the triangle inequality, however, we have

$$\sqrt{2} = d(a'', c'') \leq d(a'', b') + d(b', c'') < b_1 + (\sqrt{2} - b_1) = \sqrt{2},$$

a contradiction. □

Let now A, B, C be three disjoint unit spheres, and let $\mathcal{K}(ABC)$ be the set of vectors v such that there is an oriented line with direction vector v that intersects the spheres in the order ABC. Holmsen et al. [5] have shown that in the *three-dimensional* case, the set $\mathcal{K}(ABC)$ is convex. It is not known whether this is true in higher dimensions, but we can easily use their result to prove something weaker.

Lemma 7 *If A,B,C are disjoint unit spheres in $\mathbb{R}^d$, then $\mathcal{K}(ABC)$ is star-shaped with center $c - a$.*

Note that the lemma shows in particular that the set $\mathcal{K}(ABC)$ is connected, which in turn implies that the set of lines intersecting ABC in this order is a connected set in line space.

Proof. Let Π be the two-dimensional plane through a, b, and c. If no line intersects ABC in this order, then $\mathcal{K}(ABC)$ is empty and the lemma holds. Otherwise, B intersects the convex hull of A and C, and there is a line $\ell_0 \subset \Pi$ with direction vector $c - a$ intersecting ABC in this order. Let now ℓ be an arbitrary oriented line intersecting ABC in this order.

First, assume that ℓ is not parallel to Π. Let v be the direction vector of ℓ, and Λ be the subspace spanned by Π and v. Let Π' be a hyperplane orthogonal to ℓ, and let a', b', and c' be the orthogonal projection of a, b, c on Π'. We have $a' = a + \lambda v$ for some $\lambda \in \mathbb{R}$, so from $a, v \in \Lambda$ follows $a' \in \Lambda$, and analogously $b', c' \in \Lambda$. The $d-1$-dimensional unit spheres in Π' with centers a', b', and c' have the point $\ell \cap \Pi'$ in common. That implies that the circumcircle of the triangle $a'b'c'$ has radius at most 1. Let p be the center of this circumcircle. The line $\ell_1 = \{p + \lambda v \mid \lambda \in \mathbb{R}\}$ is parallel to ℓ and intersects ABC in this order. The 3-space Λ contains Π, and thus intersects

A,B, and C in three three-dimensional spheres of equal radii. Furthermore, it contains ℓ_0 and ℓ_1, two transversals to ABC in this order. By Holmsen et al.'s Lemma 1 [5], the set of directions of $\mathcal{K}(ABC)$ restricted to Λ is convex, so the lemma follows.

If ℓ is parallel to Π, we choose Λ as the affine hull of $\Pi \cup \ell$ and the same arguments as above work with $\ell_1 = \ell$. □

We also need the following trigonometric inequalities.

Lemma 8 *Let α, β be two angles, and let $0 \leq \xi, \zeta \leq 1$. Then*

$$2\xi\zeta\cos(\beta - \alpha) \leq \xi^2 + \zeta^2 - (\xi\cos\alpha - \zeta\cos\beta)^2,$$

where equality holds if and only if $\xi \sin\alpha = \zeta \sin\beta$.

Proof. We have

$$\begin{aligned}
0 \leq (\xi\sin\alpha - \zeta\sin\beta)^2 &= \xi^2\sin^2\alpha - 2\xi\zeta\sin\alpha\sin\beta + \zeta^2\sin^2\beta \\
&= \xi^2 - \xi^2\cos^2\alpha - 2\xi\zeta\sin\alpha\sin\beta + \zeta^2 - \zeta^2\cos^2\beta, \\
2\xi\zeta\cos(\beta - \alpha) &= 2\xi\zeta\sin\alpha\sin\beta + 2\xi\zeta\cos\alpha\cos\beta \\
&\leq \xi^2 + \zeta^2 - \xi^2\cos^2\alpha + 2\xi\zeta\cos\alpha\cos\beta - \zeta^2\cos^2\beta, \\
&= \xi^2 + \zeta^2 - (\xi\cos\alpha - \zeta\cos\beta)^2.
\end{aligned}$$

All inequalities are equalities if and only if $\xi\sin\alpha - \zeta\sin\beta = 0$. □

Corollary 9 *Let α, β be angles. Then*

$$2\cos(\alpha + \beta) \geq (\sin\alpha - \sin\beta)^2 - 2.$$

Proof. Let $\alpha' := \alpha - \pi/2$, $\beta' := \pi/2 - \beta$, and apply Lemma 8 with $\xi = \zeta = 1$. □

The following lemma is our first major geometric result. It settles a conjecture by Holmsen et al. [5].

Lemma 10 *Given three disjoint unit spheres A, B, and C in $\mathbb{R}^d$, and a directed line ℓ with direction vector v stabbing them in the order ABC. Then*

$$\angle(v, c - a) < \pi/4.$$

The bound $\pi/4$ is tight, as can be seen by chosing abc to be a nearly rectangular triangle. If one wishes to bound the angle between v and the *plane* spanned by a, b, c, then the maximal angle ϑ is given by $1/\cos\vartheta = \sqrt{9 + 6\sqrt{3}}/3$, which is roughly 43° [11].

Proof. By Lemma 7, if $\angle(v, c - a) \geq \pi/4$ then there is also a line transversal with angle exactly $\pi/4$. Thus, we assume that $\angle(v, c - a) = \pi/4$. We choose a coordinate system where the line ca is the x_1-axis, $v = (-1, -1, 0, \ldots)$ and the line ℓ goes through the point $(0, 0, \rho, 0, \ldots)$. The x_1-axis intersects the cylinder with axis ℓ and radius 1 in the segment from $x_1 = -\sqrt{2(1 - \rho^2)}$ to $x_1 = \sqrt{2(1 - \rho^2)}$. This implies $-\sqrt{2(1 - \rho^2)} \leq c_1 < 0 < a_1 \leq \sqrt{2(1 - \rho^2)}$. Without loss of generality, we can assume $b_1 \geq 0$ (otherwise we exchange the role of a and c). Let a', b', c' be the points on ℓ closest to a, b, and c, and let Λ be the three-dimensional subspace spanned by the $x_1x_2x_3$-axes. Since $\ell \subset \Lambda$, we have $a', b', c' \in \Lambda$; and clearly $a, c \in \Lambda$ as these points lie on the x_1-axis. The point b, however, may not be in Λ; let b'' be the orthogonal projection of b on Λ, so that the vector $b - b''$ is orthogonal to any vector in Λ.

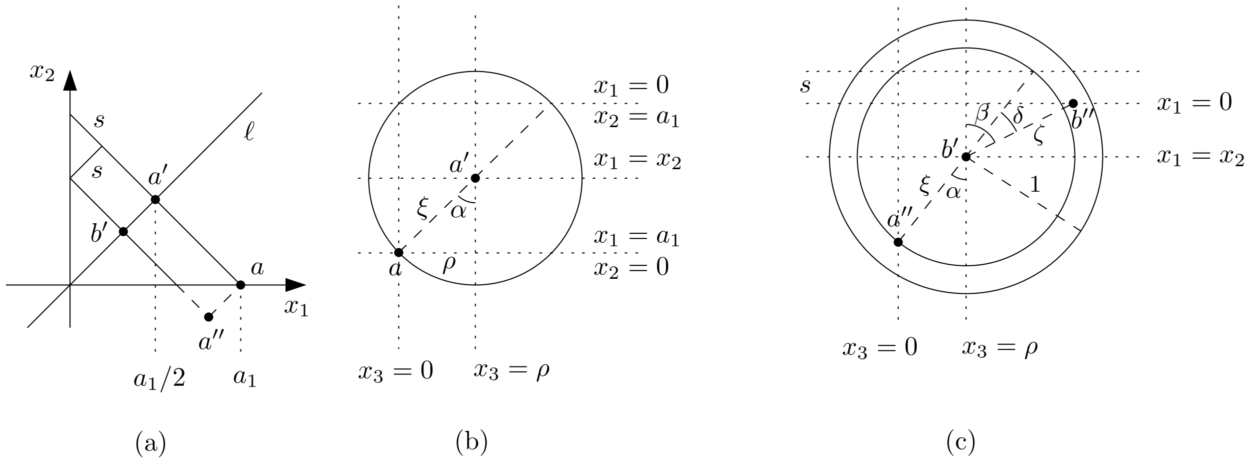

Figure 3: Illustration for Lemma 10.

Let $s := d(a', b')$, $\xi := d(a', a)$, and $\zeta := d(b', b'')$. The points a', b', and c' appear in that order along ℓ since ℓ meets the spheres in the order ABC. We therefore have $0 \le s \le d(a', c')$, and in fact even $0 < s < d(a', c')$ as ℓ may not be tangent to two spheres in the same point. Let Γ_a, Γ_b be the (two-dimensional) planes orthogonal to ℓ in Λ and containing the points a' and b', respectively. By definition of b' the point b lies in the hyperplane orthogonal to ℓ passing through b', and b'' is thus in Γ_b. Let a'' be the projection of a on Γ_b, such that $a'' - b' = a - a'$. Figure 3 illustrates the situation: (a) is a projection on the x_1x_2-plane, (b) shows the plane Γ_a, and (c) shows the plane Γ_b. We have

$$1 \ge d(b', b)^2 = d(b', b'')^2 + d(b'', b)^2 = \zeta^2 + d(b'', b)^2,$$

and so $d(b'', b)^2 \le 1 - \zeta^2$. On the other hand, we have ($a - a''$ is orthogonal to $b'' - a''$, and $b'' - b$ is orthogonal to $a - b''$)

$$4 \le d(a, b)^2 = d(a, a'')^2 + d(a'', b'')^2 + d(b'', b)^2 \le s^2 + d(a'', b'')^2 + 1 - \zeta^2.$$

It remains to bound $d(a'', b'')$. We apply the cosine-theorem on the triangle $a''b'b''$:

$$d(a'', b'')^2 = d(a'', b')^2 + d(b', b'')^2 + 2d(a'', b')d(b', b'') \cos\delta = \xi^2 + \zeta^2 + 2\xi\zeta \cos\delta,$$

where $\delta := \angle(b'' - b', b' - a'')$. Altogether, we have

$$4 \le s^2 + \xi^2 + \zeta^2 + 2\xi\zeta\cos\delta + 1 - \zeta^2 \le s^2 + \xi^2 + 1 + 2\xi\zeta\cos\delta. \tag{1}$$

If one keeps all other values fixed, the right hand side becomes maximal when δ is as small as possible. Since $b_1 \ge 0$, δ is minimal when b'' is in the plane $x_1 = 0$. In that case, $\delta = \beta - \alpha$, where $\beta = \angle(b'' - b', u)$, $\alpha = \angle(b' - a'', u)$ and $u = (-1, 1, 0)$. In this situation we have

$$s = \xi\cos\alpha - \zeta\cos\beta,$$

see Figure 3 (c). By Lemma 8 we have

$$2\xi\zeta\cos\delta = 2\xi\zeta\cos(\beta - \alpha) \le \xi^2 + \zeta^2 - (\xi\cos\alpha - \zeta\cos\beta)^2 = \xi^2 + \zeta^2 - s^2. \tag{2}$$

Combining this with Inequ. (1), we obtain $3 \le 2\xi^2 + \zeta^2$. Since $\zeta, \xi \le 1$, this implies $\xi = \zeta = 1$ and that equality holds in Inequ. (2). By Lemma 8 that implies $\sin\alpha = \sin\beta$, so either $\beta = \alpha$, or $\alpha + \beta = \pi$. In the first case $s = 0$, in the latter, $s = d(a', c')$, a contradiction. ◻

The previous angular inequality yields a first incompatible pair:

Lemma 11 *The geometric permutations ABCD and ADCB are incompatible for disjoint unit spheres.*

Proof. Let ℓ be a transversal with direction vector v stabbing four spheres in the order $ABCD$, and let ℓ' be a transversal with direction vector v' stabbing them in the order $ADCB$. By Lemma 10, it follows that $\angle(v, d-b) < \pi/4$, and $\angle(v', b-d) < \pi/4$, and therefore $\angle(v, v') > \pi/2$. On the other hand, $\angle(v, c-a) < \pi/4$ and $\angle(v', c-a) < \pi/4$, a contradiction. □

4 The geometric permutations $ABCD$ and $BADC$ are incompatible

We start with a preparatory lemma.

Lemma 12 *Given two disjoint unit spheres A and B with centers a and b in $\mathbb{R}^d$, and a line ℓ transversing both spheres. Let p be the point of intersection of ℓ and $\Pi(A, B)$, and let q be the point on ℓ closest to b. Let δ be the angle between the line bq and the (two-dimensional) plane Γ containing ℓ and being parallel to the line ab. Then $\delta < \pi/2$ and $d(p, q) \geq \sin\delta$.*

Note that Γ is not well defined when ℓ is parallel to ab. In that case, $d(p, q) \geq 1$, and the lemma holds for any angle δ.

Proof. We choose a coordinate system where $a = (-\rho, 0, \dots, 0)$, $b = (\rho, 0, \dots, 0)$, where $\rho \geq 1$, and ℓ is the line $(\lambda \sin\beta, \lambda\cos\beta, p_3, \dots, p_d)$. Then $\Pi(A, B)$ is the hyperplane $x_1 = 0$, $g(A, B)$ is the origin, and Γ is the plane $(x_1, x_2, p_3, \dots, p_d)$.

Let q' be the orthogonal projection of q on the x_1x_2-plane, and consider the rectangular triangle $bq'q$. We have $\angle q'bq = \delta$, as it is the angle between the line bq and the x_1x_2-plane, which is parallel to Γ. We therefore have

$$d(b, q') = d(b, q)\cos\delta \leq \cos\delta.$$

Figure 4 shows the projection of the situation on the x_1x_2-plane. Consider now the projection q'' of q' on the x_1-axis. We have $\angle q'bq'' = \beta$, and so

$$d(b, q'') = d(b, q')\cos\beta \leq \cos\delta\cos\beta.$$

It follows that

$$d(q, \Pi(A, B)) = d(q'', \Pi(A, B)) = \rho - d(b, q'') \geq 1 - \cos\delta\cos\beta.$$

Since the angle between ℓ and $\Pi(A, B)$ is β, we have

$$d(p, q) = \frac{d(q, \Pi(A, B))}{\sin\beta} \geq \frac{1 - \cos\delta\cos\beta}{\sin\beta}.$$

Finally, we observe that

$$1 \geq \cos(\beta - \delta) = \sin\delta\sin\beta + \cos\delta\cos\beta,$$

and so $1 - \cos\delta\cos\beta \geq \sin\delta\sin\beta$, and we obtain

$$d(p, q) \geq \frac{\sin\delta\sin\beta}{\sin\beta} = \sin\delta.$$

□

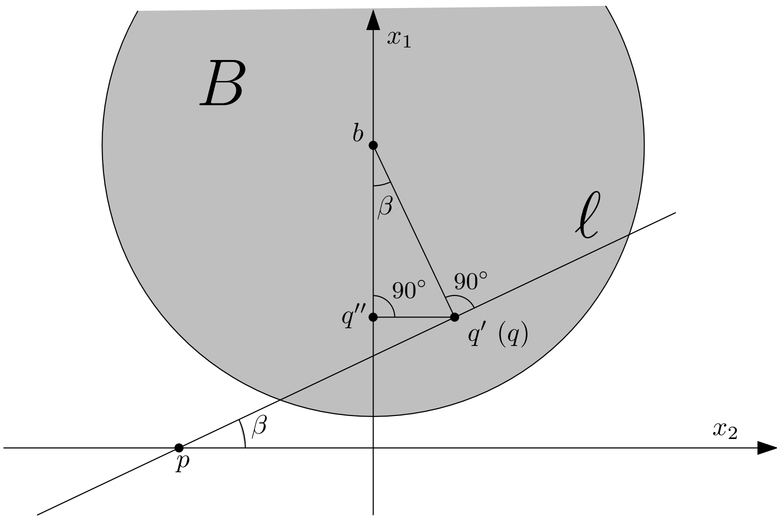

Figure 4: The situation projected on the x_1x_2-plane

We now fix four disjoint unit spheres A, B, C, D in $\mathbb{R}^d$. Let $\Pi_1 := \Pi(A, B)$, $\Pi_2 = \Pi(C, D)$, $g_1 := (a + b)/2$, and $g_2 := (c + d)/2$. Also let φ be the angle between the normals of Π_1 and Π_2.

A line transversal ℓ for the four spheres must intersect Π_1 and Π_2. We define $t(\ell)$ to be the finite segment on ℓ between the two intersection points.

Lemma 13 *Given four disjoint unit spheres A, B, C, D in $\mathbb{R}^d$ as above. Assume there is a line transversal ℓ intersecting the four spheres in the order $ABCD$, and a line transversal ℓ' intersecting them in the order $BADC$. Then*

$$\min\{|t(\ell)|, |t(\ell')|\} \leq \sin\varphi.$$

Proof. We choose a coordinate system where Π_1 is the hyperplane $x_1 = 0$, Π_2 is the hyperplane $x_1 \cos\varphi - x_2 \sin\varphi = 0$, and so the intersection $\Pi_1 \cap \Pi_2$ is the subspace $x_1 = x_2 = 0$. We can make this choice such that the x_1-coordinate of a is < 0, and that the x_2-coordinate of c is less than the x_2-coordinate of d. We can also assume that the x_2-coordinate of g_1 is ≥ 0 (otherwise we swap A with B, C with D, and ℓ with ℓ'). Figure 5 shows the projection of the situation on the x_1x_2-plane.

Since ℓ stabs A before B and C before D, it intersects Π_1 from bottom to top, and Π_2 from left to right. The segment $t(\ell)$ therefore lies in the top-left quadrant of Figure 5. On the other hand, ℓ' stabs B before A and D before C, so it intersects Π_1 from top to bottom, and Π_2 from right to left, and so the segment $t(\ell')$ lies in the bottom-right quadrant of the figure.

We introduce some further notation: Let $t := |t(\ell)|$, $t' := |t(\ell')|$, let $p_i := \ell \cap \Pi_i$, $p'_i := \ell' \cap \Pi_i$, let β_i be the angle between ℓ and Π_i, and let β'_i be the angle between ℓ' and Π_i. Let u_1 (u'_1) be the orthogonal projection of p_1 (p'_1) on Π_2, u_2 (u'_2) the orthogonal projection of p_2 (p'_2) on Π_1. Consider the rectangular triangle $p_1u_2p_2$. We have $\angle u_2p_1p_2 = \beta_1$, and so

$$t \sin\beta_1 = d(p_2, u_2) = d(p_2, \Pi_1). \tag{3}$$

Similarly, we can consider the rectangular triangles $p_2u_1p_1$, $p'_1u'_2p'_2$, and $p'_2u'_1p'_1$ to obtain

$$\begin{aligned} t \sin\beta_2 &= d(p_1, u_1) = d(p_1, \Pi_2), & (4)\\ t' \sin\beta'_1 &= d(p'_2, u'_2) = d(p'_2, \Pi_1), & (5)\\ t' \sin\beta'_2 &= d(p'_1, u'_1) = d(p'_1, \Pi_2). & (6) \end{aligned}$$

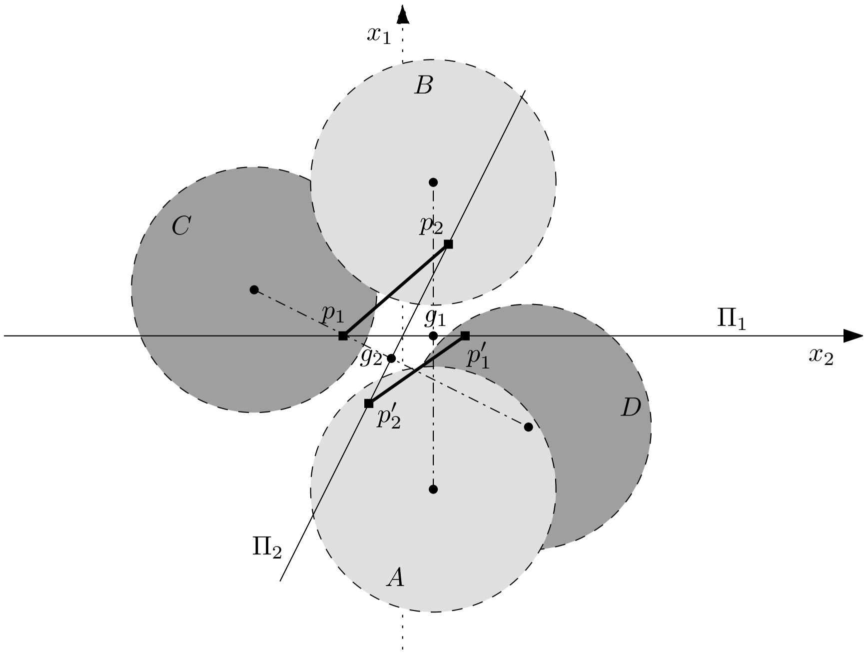

Figure 5: The two hyperplanes define four quadrants

We now distinguish two cases.

The *first case* occurs if, as in the figure, the x_1-coordinate of g_2 is ≤ 0. By Lemma 5 we have $d(p_2, g_2) \leq \sin\beta_2$. Since p_2 and g_2 lie on opposite sides of Π_1, we have $d(p_2, \Pi_1) \leq \sin\beta_2 \sin\varphi$. Similarly, we have $d(p_1, g_1) \leq \sin\beta_1$, and p_1 and g_1 lie on opposite sides of Π_2, implying $d(p_1, \Pi_2) \leq \sin\beta_1 \sin\varphi$. Plugging into Eq. (3) and (4), we obtain

$$t \leq \min\{\frac{\sin\beta_2}{\sin\beta_1}, \frac{\sin\beta_1}{\sin\beta_2}\} \sin\varphi \leq \sin\varphi,$$

which proves the lemma for this case.

The *second case* occurs if the x_1-coordinate of g_2 is > 0. We let $s_1 := d(g_1, \Pi_2)$, and $s_2 := d(g_2, \Pi_1)$. Applying Lemma 5 , we then have

$$\begin{aligned} d(p_2, \Pi_1) &\leq d(p_2, g_2)\sin\varphi + s_2 \leq \sin\beta_2 \sin\varphi + s_2, && (7)\\ d(p_1, \Pi_2) &\leq d(p_1, g_1)\sin\varphi - s_1 \leq \sin\beta_1 \sin\varphi - s_1, && (8)\\ d(p_2', \Pi_1) &\leq d(p_2', g_2)\sin\varphi - s_2 \leq \sin\beta_2' \sin\varphi - s_2, && (9)\\ d(p_1', \Pi_2) &\leq d(p_1', g_1)\sin\varphi + s_1 \leq \sin\beta_1' \sin\varphi + s_1. && (10) \end{aligned}$$

Plugging Ineqs. (7) to (10) into (3) to (6), we obtain

$$t \leq \frac{\sin\beta_2 \sin\varphi + s_2}{\sin\beta_1}, \quad (11)$$

$$t \leq \frac{\sin\beta_1 \sin\varphi - s_1}{\sin\beta_2}, \quad (12)$$

$$t' \leq \frac{\sin\beta_2' \sin\varphi - s_2}{\sin\beta_1'}, \quad (13)$$

$$t' \leq \frac{\sin\beta_1' \sin\varphi + s_1}{\sin\beta_2'}. \quad (14)$$

We want to prove that $\min(t, t') \leq \sin\varphi$. We assume the contrary. From $t > \sin\varphi$ and Ineq. (12)

we obtain

$$\sin\beta_2 \sin\varphi < \sin\beta_1 \sin\varphi - s_1,$$

and from $t' > \sin\varphi$ and Ineq. (13) we get

$$\sin\beta_1' \sin\varphi < \sin\beta_2' \sin\varphi - s_2.$$

Plugging this into Ineq. (11) and (14) results in

$$\begin{aligned} t &\leq \frac{\sin\beta_2 \sin\varphi + s_2}{\sin\beta_1} < \frac{\sin\beta_1 \sin\varphi - s_1 + s_2}{\sin\beta_1} = \sin\varphi + \frac{s_2 - s_1}{\sin\beta_1}, \\ t' &\leq \frac{\sin\beta_1' \sin\varphi + s_1}{\sin\beta_2'} < \frac{\sin\beta_2' \sin\varphi - s_2 + s_1}{\sin\beta_2'} = \sin\varphi + \frac{s_1 - s_2}{\sin\beta_2'}. \end{aligned}$$

It follows that if $s_2 < s_1$ then $t < \sin\varphi$, otherwise $t' < \sin\varphi$. In either case the lemma follows. □

Theorem 14 *The geometric permutations ABCD and BADC are incompatible for disjoint unit spheres in* $\mathbb{R}^d$.

Proof. Assume two line transversals ℓ and ℓ' exist, realizing the geometric permutations *ABCD* and *BADC*. By Lemma 13 we have $\min\{|t(\ell)|, |t(\ell')|\} \leq \sin\varphi$. Without loss of generality, we can assume that $|t(\ell)| \leq \sin\varphi$.

Let n_i be the unit normal vector of Π_i pointing into the halfspace containing $t(\ell)$, for $i = 1, 2$. We can express n_i uniquely as $n_i = u_i + \lambda_i v$, where v is the direction vector of ℓ and $u_i v = 0$. Notice that $\|u_i\| \leq \|v_i\| = 1$. Since ℓ stabs A before B, we have $n_1 v > 0$. Since it stabs C before D, we have $n_2 v < 0$. This implies $\lambda_1 > 0$, $\lambda_2 < 0$, and therefore $\lambda_1\lambda_2 < 0$. Recall that $\varphi = \angle(n_1, n_2)$, and let $\vartheta = \angle(u_1, u_2)$. We have

$$\cos\varphi = n_1 n_2 = (u_1 + \lambda_1 v)(u_2 + \lambda_2 v) = u_1 u_2 + \lambda_1\lambda_2 v^2 < u_1 u_2 < \frac{u_1 u_2}{\|u_1\|\|u_2\|} = \cos\vartheta,$$

and so $\vartheta < \varphi$.

Let $p_i = \ell \cap \Pi_i$, for $i = 1, 2$, let $q_1 \in \ell$ be the point closest to b, and let $q_2 \in \ell$ be the point closest to c. The points q_1 and q_2 lie between p_1 and p_2, that is, in the segment $t(\ell)$, and so we have

$$d(p_1, q_1) + d(q_1, q_2) + d(q_2, p_2) = d(p_1, p_2) = |t(\ell)| \leq \sin\varphi, \tag{15}$$

the last inequality stemming from Lemma 13.

Let δ_1 be the angle between u_1 and $b - q_1$, and let δ_2 be the angle between u_2 and $c - q_2$. Let Γ_1 be the (two-dimensional) plane containing ℓ and being parallel to the line ab. The vectors v and u_1 form an orthogonal basis for Γ_1. We can uniquely express $b - q_1 = \lambda u_1 + \mu v + w$ with $w \cdot u_1 = w \cdot v = 0$. Since

$$0 = (b - q_1) \cdot v = (\lambda u_1 + \mu v + w) \cdot v = \lambda u_1 \cdot v + \mu v \cdot v + w \cdot v = \mu \|v\|^2,$$

we have $\mu = 0$ and so the angle between Γ_1 and the line bq_1 is identical to the angle $\angle(u_1, b - q_1) = \delta_1$. By Lemma 12, this implies that $d(p_1, q_1) \geq \sin\delta_1$. Completely analogously, we prove that δ_2 is identical to the angle between the line cq_2 and the (two-dimensional) plane Γ_2 that contains ℓ and is parallel to the line cd. By Lemma 12, this implies that $d(p_2, q_2) \geq \sin\delta_2$. Applying Ineq. (15) results in

$$\sin\delta_1 + \sin\delta_2 + d(q_1, q_2) \leq \sin\varphi. \tag{16}$$

Consider the hyperplane Γ orthogonal to ℓ in q_1. It contains the points q_1 and b, and its normal is v. Let c' be the orthogonal projection of c on Γ, so that we have $c - q_2 = c' - q_1$. Let $\psi := \angle c' q_1 b$. Since B and C are disjoint, we have

$$4 \leq d(b, c)^2 = d(q_1, q_2)^2 + d(b, c')^2 \tag{17}$$

Consider now the triangle bq_1c'. By the cosine-theorem, we have

$$\begin{aligned} d(b,c')^2 &= d(b,q_1)^2 + d(c',q_1)^2 - 2d(b,q_1)d(c',q_1)\cos\psi \\ &= d(b,q_1)^2 + d(c,q_2)^2 - 2d(b,q_1)d(c,q_2)\cos\psi \\ &\leq 2 - 2d(b,q_1)d(c,q_2)\cos\psi. \end{aligned}$$

Since $\delta_1, \delta_2 < \pi/2$, Ineq. (16) implies $d(q_1,q_2) \leq 1$. Combining with Ineq. (17) results in $d(b,c')^2 \geq 3$, which implies $\cos\psi < 0$. We can therefore apply the upper bounds $d(b,q_1) \leq 1$ and $d(c,q_2) \leq 1$ again to obtain $d(b,c')^2 \leq 2 - 2\cos\psi$. Together with Ineq. (17) this gives $4 \leq d(q_1,q_2)^2 + 2 - 2\cos\psi$, or

$$2\cos\psi \leq d(q_1,q_2)^2 - 2. \tag{18}$$

Let now $\delta := \delta_1 + \delta_2$. We claim that $\delta \leq \pi/2$. Indeed, assume that $\delta > \pi/2$. By Ineq. (16), we have

$$\sin\delta_1 + \sin(\delta - \delta_1) = \sin\delta_1 + \sin\delta_2 \leq \sin\varphi \leq 1.$$

The function $\delta_1 \mapsto \sin\delta_1 + \sin(\delta - \delta_1)$ over the interval $[\delta - \pi/2, \pi/2]$ is minimized for $\delta_1 = \pi/2$ or $\delta_1 = \delta - \pi/2$, where its value is $\sin\pi/2 + \sin(\delta - \pi/2) > 1$, a contradiction.

We now argue that $\varphi + \delta \leq \pi$. This is true if $\varphi \leq \pi/2$. Otherwise, $\pi - \varphi < \pi/2$. By Ineq. (16) we have

$$\sin\delta \leq \sin\delta_1 + \sin\delta_2 \leq \sin\varphi = \sin(\pi - \varphi),$$

which implies $\delta \leq \pi - \varphi$ and therefore $\delta + \varphi \leq \pi$. Since $\vartheta < \varphi$, this also implies $\vartheta + \delta < \pi$.

Consider now the angle $\psi = \angle bq_1c'$. We can write it as the sum of the three *oriented* angles $\angle(b - q_1, u_1)$, $\angle(u_1, u_2)$, and $\angle(u_2, c' - q_1)$. Since $\vartheta + \delta_1 + \delta_2 \leq \pi$, this implies $0 \leq \psi \leq \vartheta + \delta_1 + \delta_2 = \vartheta + \delta < \varphi + \delta \leq \pi$. We apply Corollary 9 and obtain

$$2\cos\psi > 2\cos(\varphi + \delta) \geq (\sin\varphi - \sin\delta)^2 - 2.$$

Together with Ineq. (18) we get $(\sin\varphi - \sin\delta)^2 < d(q_1,q_2)^2$, so $d(q_1,q_2) > \sin\varphi - \sin\delta$. Combining with Ineq. (16), we obtain

$$\sin\varphi = \sin\delta + \sin\varphi - \sin\delta < \sin\delta_1 + \sin\delta_2 + d(q_1,q_2) \leq \sin\varphi,$$

a contradiction. □

5 Putting everything together

We now apply the combinatorial results of Section 2 to our geometric results. Lemma 2 immediately implies the following theorem, using Lemma 11 and Theorem 14.

Theorem 15 *Let $\mathcal{S}$ be a family of disjoint unit spheres in $\mathbb{R}^d$. Then $\mathcal{S}$ admits at most three distinct geometric permutations.*

This is the first bound valid for a small number of spheres in dimension greater than 2. To improve the bound to the optimal 2, we need the two additional incompatible pairs of Lemma 3. Our proof of incompatibility of these pairs, however, uses the additional assumption that $n \geq 9$. Note that this threshold is independant of the dimension.

Lemma 16 *Let $\mathcal{S}$ be a family of $n \geq 9$ disjoint unit spheres in $\mathbb{R}^d$. Then any two line transversals for $\mathcal{S}$ make an angle of less than $\pi/4$.*

Proof. Let ℓ and ℓ' be two line transversals for $\mathcal{S}$, and let $\mathcal{C}$ and $\mathcal{C}'$ be cylinders of radius 1 with axis ℓ and ℓ', respectively. The centers of all spheres in $\mathcal{S}$ are contained in $\mathcal{C} \cap \mathcal{C}'$. If ℓ and ℓ' make an angle of at least $\pi/4$, then $\mathcal{C} \cap \mathcal{C}'$ is contained in a section of $\mathcal{C}$ of length at most $2+2\sqrt{2} < 4\sqrt{2}$. By Lemma 6, this implies $n \leq 8$, a contradiction. □

The threshold 9 can probably be lowered by analyzing the shape of $\mathcal{C} \cap \mathcal{C}'$ more carefully. We do not pursue this, as values of n remain where our best bound on the number of geometric permutations is 3.

We can now prove that $(ABCD, ADBC)$ and $(ABCD, CADB)$ are incompatible pairs.

Lemma 17 *Let $\mathcal{S}$ be a family of $n \geq 9$ disjoint unit spheres in $\mathbb{R}^d$. Then the pairs $(ABCD, ADBC)$ and $(ABCD, CADB)$ are incompatible for $\mathcal{S}$.*

Proof. Let v be the direction vector of a line transversal realizing $ABCD$, and let v' be the direction vector of a transversal realizing either $ADBC$ or $CADB$. By Lemma 10, $\angle(v, d - b) < \pi/4$. On the other hand, $\angle(v', b - d) < \pi/2$, and so $\angle(v, v') > \pi/4$, a contradiction with Lemma 16. □

The final theorem now follows from Lemma 3, using Lemmas 11 and 17 and Theorem 14.

Theorem 18 *Let $\mathcal{S}$ be a family of $n \geq 9$ disjoint unit spheres in $\mathbb{R}^d$. Then $\mathcal{S}$ admits at most two distinct geometric permutations, which differ only in the swapping of two adjacent spheres.*

Our results also improve the constants involved in recent results by Holmsen et al. [5]. First, Lemma 10 implies the following improvement to Holmsen et al.'s Theorem 2, a Hadwiger-type theorem (their constant is 12).

Theorem 19 *Let $\mathcal{S}$ be a family of disjoint unit spheres in $\mathbb{R}^3$. If there is a linear ordering on $\mathcal{S}$ such that every 9 members are met by a directed line consistent with that ordering, then $\mathcal{S}$ admits a line transversal.*

This improvement, combined with Theorem 18, reduces the constant in their Helly-type Theorem 1 from 46 to 18. (The justification for both improvements can be found in Holmsen et al.'s paper [5], in the first remark of their Section 4.)

Theorem 20 *Let $\mathcal{S}$ be a family of n disjoint unit spheres in $\mathbb{R}^3$. There exists an integer $n_0 \leq 18$ such that if any subset $\mathcal{S}' \subset \mathcal{S}$ of size at most n_0 admits a line transversal, then $\mathcal{S}$ admits a line transversal.*

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